AN ABSTRACT OF THE DISSERTATION OF

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The purpose of this dissertation is to develop and apply methods using the division algebra of the octonions in mathematical physics. This investigation, built upon the foundation of the theory of Clifford algebras, is motivated by the correspondence between supersymmetric theories and division algebras.

We extend the theory of representations of Clifford algebras to octonions. This extension is complicated by the non-associativity of the octonions. However, the alternative property of the octonions is shown to be sufficient to overcome this difficulty. The effects of the choice of an octonionic multiplication rule are found to be related to a change of basis on the carrier space of a representation. Octonionic conjugation and matrix transposition of a representation is seen to induce a representation based on the opposite octonionic algebra. We describe octonionic representations for Clifford algebras over spaces of 6, 7, 8, and 9+1 dimensions. These representations are used to give octonionic descriptions of generating sets of the Clifford groups and the orthogonal groups in these dimensions. A similar description for the exceptional Lie group G_2, the automorphism group of the octonionic algebra, is found.

The octonionic description of the vector and spin representations of $SO(8)$ are combined to give a unified picture of the triality automorphisms of $SO(8)$ which manifestly shows their $\Sigma_3 \times SO(8)$ structure and unequivocally displays the symmetry interchanging the spaces of vectors, even spinors, and odd spinors.

These octonionic methods are then applied to the Casalbuoni-Brink-Schwarz superparticle for which we rederive the general classical solution of the equations of motion. We introduce a superspace variable containing both the bosonic and fermionic degrees of freedom as a 3×3 Grassmann, octonionic, Jordan matrix. We succeed in giving a unified description of supersymmetry and Lorentz transformations exclusively involving Jordan products of such 3×3 matrices.

The results of this dissertation provide a basis for the further investigation of supersymmetry using the octonionic algebra. In particular, we conjecture that an extension of the treatment of the superparticle to the Green-Schwarz superstring is possible. Such an extension may provide a useful tool to use in the covariant quantization of the superstring.

Octonions and Supersymmetry

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Jörg Schray, Author

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soli deo gloria

CONTRIBUTION OF AUTHORS

Dr. Manogue was involved, through discussions and comments, in all of the presented work. Section 3.5 is almost entirely due to her and predates the rest of the work in this dissertation. The remaining sections of chapter 3 as well as chapters 2 and 4 are J. Schray's original work except as referenced.

TABLE OF CONTENTS

LIST OF FIGURES

OCTONIONS AND SUPERSYMMETRY

1. INTRODUCTION

The discovery of the octonions by Cayley [17] and Graves [31,63] followed soon after Hamilton's [60] discovery of quaternions in the middle of the 19th century. There was some speculation that the octonions could be extended further, but a celebrated theorem by Hurwitz [67] concluded that the sequence of normed division algebras over the real numbers contains only the reals, $\mathbb{R}$, themselves; the complexes, $\mathbb{C}$; the quaternions, $\mathbb{H}$; and the octonions, $\mathbb{O}$. (We also denote these algebras by $\mathbb{K}_n$, where $n = 1, 2, 4,$ and 8 is their respective dimension as vector spaces over $\mathbb{R}$.) So, the octonions have a special status as the normed division algebra of highest dimension. But what is their mathematical and physical significance?

For the quaternions, Hamilton [62,61] himself pursued this question and found wide applications in geometry and celestial mechanics. These applications rely on the identification of vectors in 3 dimensions with purely imaginary quaternions. Scalar and vector products are recovered as the negative of the real and as the imaginary part, respectively, of the product of the corresponding quaternions. The octonions on the other hand did not receive much attention.

In Cartan's [13,98] classification of simple Lie groups, we have three infinite families of groups related to the reals, complexes, and quaternions, namely the simple orthogonal, the simple unitary and the symplectic groups. What then are the Lie groups corresponding to the octonions? Apart from the three infinite families, there are five exceptional Lie groups. The lowest dimensional exceptional Lie group,

G_2, is the automorphism group of the octonions. The octonions are also connected to the exceptional group of next higher dimension, F_4, which is the automorphism group of the exceptional Jordan algebra [69–72] of 3×3 octonionic hermitian matrices. Of course, the non-exceptional Jordan algebras are just matrix algebras over the reals and complexes [68]. (Since quaternions have a matrix representation, they do not give rise to additional Jordan algebras.)

This pattern of the octonions being related to exceptional structures repeats itself for the geometries of spheres. The sequence of Hopf [65,66] maps

$$S^{2n-1} \rightarrow S^n \qquad (n = 1, 2, 4, 8), \tag{1.1}$$

are the only existing sphere fibrations. The fact that the sequence of possible dimensions coincides with the sequence of the division algebras is not an accident. The fibres of these projections are the only parallelizable spheres, S^0, S^1, S^3, and S^7 [2,3,15,78], and, as it is pointed out below and in section 4.3, may be identified as the elements of unit norm of the respective division algebras. Whereas the unit sphere for the reals is $S^0 \cong O(1)$, for the complexes $S^1 \cong U(1)$, and for the quaternions $S^3 \cong SO(3)$, the octonionic unit sphere, S^7, is the only parallelizable sphere which does not allow a group structure [22]. So even within this exceptional sequence, the octonions play a special role and are definitely worth studying from a mathematical point of view.

Physicists became interested in the octonions with the rise of supersymmetric theories (including superstrings and supergravity). Supersymmetry [40,81,90,94] is a postulated symmetry between matter and forces, fermionic and bosonic degrees of freedom. Such a symmetry implies the existence of supersymmetric partner particles for all existing elementary particles, for example there should be a spin $\frac{3}{2}$ fermion, corresponding to the spin 1 photon. However, such particles are not observed. Why then is such a theory considered? A possible explanation for the lack of evidence for this symmetry is that the ground state of the physical world, i.e., the vacuum state, may not exhibit this symmetry even though the dynamics, i.e., the interactions, do. In this case the symmetry is said to be broken and could only be recovered at sufficiently high energies. Moreover, among field theoretical models only a supersymmetric theory has the prospect of unifying the gravitational interaction with the electroweak and strong interaction [94], since the mediating particles have differing spins. Such a unification of interactions would incorporate a unification of general relativity and quantum field theory, which are the cornerstones of our modern understanding of the physical world. The unification of these two theories becomes necessary when we extrapolate their basic claims to the energy E_{Planck} of the Planck region, which is

$$E_{\text{Planck}} = c^2\sqrt{\frac{\hbar c}{G}} \approx 10^{19}\ \text{GeV}. \tag{1.2}$$

At such energies the Schwarzschild radius of a particle is of the same order of magnitude as its Compton wavelength, i.e., the length scale of the curvature of space of general relativity and the length scale corresponding to Heisenberg's uncertainty principle become comparable. Conventional attempts to quantize gravity fail because the resulting field theory is not renormalisable, i.e., infinities arising from self-interactions cannot be eliminated. Supersymmetric theories are more promising in this regard, since cancellations between bosonic and fermionic terms make these theories remarkably well-behaved. Therefore, supersymmetric theories are deemed to be among the best candidates for extending existing theories, despite the lack of supporting experimental evidence.

How then are supersymmetric theories connected with the division algebras? A class of supersymmetric theories is seen [5,38,74] to rely on a parametrization of lightlike vectors in terms of a spinorial variable ψ:

$$y_\mu = \overline{\psi}\gamma_\mu\psi, \qquad y_\mu y^\mu = (\overline{\psi}\gamma_\mu\psi)(\overline{\psi}\gamma^\mu\psi) = 0, \tag{1.3}$$

where the γ_μ are the Dirac matrices. (ψ is also called a twistor [83].) Normalizing the components of ψ, the map

$$\psi \mapsto \overline{\psi}\gamma_\mu\psi \tag{1.4}$$

becomes a sphere fibration, which can also be expressed in terms of the division algebras [21]. Actually a generalization (4.30) of the spinor identity (1.3) is needed for this class of supersymmetric theories.

Corresponding to the Hopf maps, there exist descriptions of the Lorentz groups in terms of the division algebras in the relevant dimensions:

$$SL(2,\mathbb{K}_n) \cong SO(n+1,1), \tag{1.5}$$

i.e., Lorentz transformations act on vectors, represented by 2×2 hermitian matrices over $\mathbb{K}_n$, via the simple linear group $SL(2,\mathbb{K}_n)$. For the octonions, only the Lie algebra version [24,96] of these transformations was understood prior to this work. The first contribution of this dissertation is the explicit demonstration of (1.5) on the group level, which is found in chapter 3. Using a constructive approach, this chapter examines octonionic representations of $SO(7)$, $SO(8)$, and $SO(9,1)$, paralleled by quaternionic representations of $SO(3)$, $SO(4)$, and $SO(5,1)$. We also find a remarkable octonionic description for a generating set of the exceptional Lie group, G_2, which exhibits a structure similar to the orthogonal groups.

Chapter 2 establishes a theoretical framework for these constructions by extending the theory of representations of Clifford algebras to octonions. This extension is complicated by the non-associativity of this division algebra. Remarkably, the alternative property of the octonions is sufficient to overcome this difficulty. We describe octonionic representations for Clifford algebras over spaces of 6, 7, 8 and 9+1 dimensions. These representations are used to give octonionic descriptions of generating sets of the Clifford groups and the orthogonal groups in these dimensions. We also observe features that are peculiar to octonionic representations: The effects of the choice of an octonionic multiplication rule is found to be related to a change of basis on the carrier space of a representation. Due to the non-commutativity of the octonions, octonionic conjugation and matrix transposition of a representation is seen to induce a representation based on the opposite octonionic algebra.

Chapter 2 culminates in an octonionic description of the triality automorphisms of $SO(8)$, which manifestly shows their $\Sigma_3 \times SO(8)$ structure and unequivocally displays the symmetry interchanging the spaces of vectors, even spinors, and odd spinors. The octonionic description of the vector and spin representations of $SO(8)$ are combined to give this unified picture. As is evident from our description, the triality symmetry is a prototype for supersymmetry and is closely related to the exceptional Jordan algebra.

A variation of the exceptional Jordan algebra involving anticommuting parameters appears in chapter 4. In this chapter our octonionic methods are applied to the Casalbuoni-Brink-Schwarz superparticle, for which we rederive the general classical solution of the equations of motion. We introduce a superspace variable containing both the bosonic and fermionic degrees of freedom as a 3×3 Grassmann, octonionic, Jordan matrix. We succeed in giving a unified description of supersymmetry and Lorentz transformations exclusively involving Jordan products of such 3×3 matrices.

Supersymmetric theories involving octonions and the exceptional Jordan algebras have also been widely explored. Among these supersymmetric theories are superparticle, twistor and superstring models; supergravity; and super Yang-Mills theories [18–21,27,28,39,43,44,52,53,55–59,73,77,82,93,97].

Chapter 5 suggests avenues for further investigations.

2. OCTONIONIC REPRESENTATIONS OF CLIFFORD ALGEBRAS AND TRIALITY

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The theory of representations of Clifford algebras is extended to employ the division algebra of the octonions or Cayley numbers. In particular, questions that arise from the non-associativity and non-commutativity of this division algebra are answered. Octonionic representations for Clifford algebras lead to a notion of octonionic spinors and are used to give octonionic representations of the respective orthogonal groups. Finally, the triality automorphisms are shown to exhibit a manifest $\Sigma_3 \times SO(8)$ structure in this framework.

2.1. INTRODUCTION

The existence of classical supersymmetric string theories in $(n+1,1)$ dimensions has been linked to the existence of the normed division algebras $\mathbb{K}_n$ [1,2], where $\mathbb{K}_n = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ for $n = 1$, 2, 4, and 8 are the algebras of the reals, complexes, quaternions, and octonions. One reason for this correspondence is the isomorphism $sl(2,\mathbb{K}_n) \cong so(n+1,1)$ on the Lie algebra level [3]. However, because of the non-associativity of the octonions, the extension of this result to finite Lorentz transformations, i.e., on the Lie group level, for $n = 8$ has posed a problem until recently [4,5]. Nevertheless, octonionic spinors based on $sl(2,\mathbb{O})$ have been used successfully as a tool to solve and parametrize classical solutions of the superstring and superparticle [5–7].

Another link between octonions and supersymmetric theories is given by the triality [8,2,9] automorphisms of $SO(8)$, which interchange the spaces of vectors, even spinors, and odd spinors. These automorphisms are constructed using the Chevalley algebra, which combines these three spaces into a single 24-dimensional algebra, which can be extended to the exceptional Jordan algebra of 3×3 octonionic hermitian matrices. A variety of articles connect this algebra to theories of the superstring, the superparticle, and supergravity [10,11].

Division algebras are also used in the spirit of GUTs to provide a group structure that contains the known interactions [12].

The contribution of this paper is to bring these many isolated observations together and place them on the foundation of the theory of Clifford algebras. Our framework allows an elegant unified derivation of all the previous results about orthogonal groups. The octonionic triality automorphisms, for example, are completely symmetric with respect to the spaces of vectors, even spinors, and odd spinors, as they should be. We also explain new features and properties of octonionic representations of Clifford algebras related to the possible choices of different octonionic multiplication rules. We also find that not all of the common constructions from complex representations have exact analogues for octonionic representations because of the non-commutativity of the octonions. For example, the octonionic analogue of the charge conjugation operation involves the opposite octonionic algebra, without which the transformation behavior is inconsistent. However, the extra structure of two distinguished octonionic algebras may turn out to be a feature of our formalism rather than a bug.

In a previous article [4] a demonstration of the construction of $SO(7)$, $SO(8)$, $SO(9,1)$, and G_2 is given, which illustrates how the octonionic algebra works explicitly. However, in this article, we only use the general algebraic properties of the octonions, rather than rely on explicit computations involving a specific multiplication rule. This approach is taken to highlight the central role of the alternativity of the octonions in the development of our formalism. In essence, we suggest the division algebra of the octonions not as an afterthought, but as a starting point for incorporating Lorentzian symmetry and supersymmetry in supersymmetrical theories. This principle is brought to fruition in a fully octonionic description of the triality automorphisms of the Chevalley algebra.

The content of this article is organized as follows: First we give a thorough introduction to composition algebras and the division algebra of the octonions. In particular, we devote a large part of section 2.2 to the investigation of the relationship amongst different multiplication tables of the octonions. In section 2.3 we state basic concepts about Clifford algebras and their representations. We characterize the Clifford group and the orthogonal group of a vector space with a metric by generating sets. This approach turns out to be better adapted to octonionic representations than the usual Lie algebra one. Then we introduce the octonionic representation of the Clifford algebra in 8-dimensional Euclidean space in section 2.4. In section 2.5, the reductions to 7 and 6 dimensions and the extension to 9+1 dimensions are discussed. In section 2.6, we introduce an octonionic description of the Chevalley algebra and show that the triality symmetry is inherent in the octonionic description. Then, in section 2.7, we briefly explain how our results with regard to sets of finite generators of Lie groups are related to the usual description in terms of infinitesimal generators of the corresponding Lie algebra. Section 2.8 discusses our results.

2.2. THE DIVISION ALGEBRA OF THE OCTONIONS

This section lays the first part of the foundation for octonionic representations of Clifford algebras, namely it introduces the octonionic algebra. The first subsection deals with some general properties of composition algebras. A subsection introducing our convention for octonions follows. We then turn our attention to the relationship among different multiplication tables for the octonions and introduce the opposite octonionic algebra. For further information and omitted proofs see [13,14,3]. A less rigorous approach is taken in [4].

2.2.1. Composition algebras

An algebra $\mathfrak{A}$ over a field $\mathbb{F}$ is a vector space over $\mathbb{F}$ with a multiplication that is distributive and $\mathbb{F}$-linear:

$$\left.\begin{aligned} x(y+z) &= xy + xz \\ (x+y)z &= xz + yz \end{aligned}\right\} \qquad \forall\, x,y,z \in \mathfrak{A}, \tag{2.1}$$

$$(fx)y = x(fy) = f(xy) \qquad \forall\, x,y \in \mathfrak{A},\ \forall\, f \in \mathbb{F}. \tag{2.2}$$

$\mathfrak{A}$ is also assumed to have a multiplicative identity $1_{\mathfrak{A}}$.

A composition algebra $\mathfrak{A}$ over a field $\mathbb{F}$ is defined to be an algebra equipped with a non-degenerate symmetric $\mathbb{F}$-bilinear form,

$$\begin{aligned} \langle\cdot,\cdot\rangle : \mathfrak{A}\times\mathfrak{A} &\to \mathbb{F} \\ (x,y) &\mapsto \langle x,y\rangle\,, \end{aligned} \tag{2.3}$$

with the special property that it gives rise to a quadratic norm form which is compatible with multiplication in the algebra:

$$\begin{aligned} |\cdot|^2 : \mathfrak{A} &\to \mathbb{F} \\ x &\mapsto |x|^2 := \langle x,x\rangle\,, \end{aligned} \tag{2.4}$$

$$|xy|^2 = |x|^2|y|^2 \quad \forall\, x,y\in\mathfrak{A}. \tag{2.5}$$

(In the case of the octonions (2.5) is known as the eight-squares theorem, i.e., a sum of eight squares is the product of two sums of eight squares, and many applications rely on this identity.) Two main consequences can be derived (see [13]) from this essential property of composition algebras. Firstly, these algebras exhibit a weak form of associativity:

$$\left.\begin{aligned} x(xy) &= (xx)y \\ (yx)x &= y(xx) \end{aligned}\right\} \quad \forall\, x,y\in\mathfrak{A}. \tag{2.6}$$

Defining the associator as a measure of the deviation from associativity via

$$[x,y,z] := x(yz)-(xy)z, \qquad x,y,z\in\mathfrak{A}, \tag{2.7}$$

then (2.6) implies

$$[x,x,y] = [y,x,x] = 0 \qquad \forall\, x,y\in\mathfrak{A} \tag{2.8}$$

or (by polarization)

$$[x,y,z] = -[x,z,y] = -[y,x,z] \qquad \forall\, x,y,z\in\mathfrak{A}, \tag{2.9}$$

i.e., the associator is an alternating function of its arguments. This weak form of associativity is also called alternativity. (2.9) and (2.6) are equivalent, if the characteristic $\chi(\mathbb{F})$ of $\mathbb{F}$ does not equal 2, which is assumed from now on. As shown in [13] alternativity implies the so-called Moufang [15] identities,

$$\left.\begin{aligned} (xyx)z &= x(y(xz)) \\ z(xyx) &= ((zx)y)x \\ x(yz)x &= (xy)(zx) \end{aligned}\right\} \quad \forall\, x, y, z \in \mathfrak{A}, \tag{2.10}$$

which will turn out to be useful later on.

Secondly, composition algebras are endowed with an involutory antiautomorphism $\cdot^*$:

$$\begin{aligned} \cdot^* : \mathfrak{A} &\to \mathfrak{A} \\ x &\mapsto x^* := 2\,\langle 1, x\rangle - x\,, \\ (xy)^* = y^*\, x^* & \qquad \forall\, x, y \in \mathfrak{A}. \end{aligned} \tag{2.11}$$

(Obviously, we view $\mathbb{F}$ as embedded in the algebra $\mathfrak{A}$ via $\mathbb{F} \cong \mathbb{F}\,1_{\mathfrak{A}} \subseteq \mathfrak{A}$, in particular $1_{\mathfrak{A}} = 1_{\mathbb{F}} = 1$. With this identification and (2.2), multiplication with an element of $\mathbb{F}$ is commutative, i.e., $\mathbb{F} \subseteq \mathcal{Z}$, where $\mathcal{Z}$ is the center of $\mathfrak{A}$.) We observe that $\cdot^*$ is linear and fixes $\mathbb{F}$. (Note that $\langle 1, 1\rangle = 1$, since $\langle x, x\rangle = \langle x, x\rangle\langle 1, 1\rangle \quad \forall\, x \in \mathfrak{A}$.) This antiautomorphism can be shown to provide a way to express the quadratic form $|\cdot|^2$:

$$x\, x^* = x^*\, x = |x|^2 \qquad \forall\, x \in \mathfrak{A}. \tag{2.12}$$

So all elements of $\mathfrak{A}$ satisfy a quadratic equation over $\mathbb{F}$:

$$x^2 - 2\langle 1, x\rangle x + |x|^2 = 0 \qquad \forall\, x \in \mathfrak{A}. \tag{2.13}$$

Polarizing (2.12) results in an expression for the bilinear form:

$$\langle x, y\rangle = \frac{1}{2}(x\, y^* + y\, x^*) \qquad \forall\, x, y \in \mathfrak{A}. \tag{2.14}$$

We determine inverses:

$$x^{-1} = \frac{x^*}{|x|^2} \qquad \forall\, x \in \mathfrak{A}, |x|^2 \neq 0. \tag{2.15}$$

However, in order to solve a linear equation $ax = b$, we need $a^{-1}(ax) = x$. To see that we do indeed have associativity in this case, we need the following relationship,

$$6[x, y, z] = [x, [y, z]] + [y, [z, x]] + [z, [x, y]] \qquad \forall\, x, y, z \in \mathfrak{A}, \tag{2.16}$$

between the associator and the commutator

$$[x, y] := xy - yx, \qquad x, y \in \mathfrak{A}, \tag{2.17}$$

which is defined as usual. So for $\chi(\mathbb{F}) \neq 2, 3$, we see that products with elements in $\mathcal{Z}$ are associative:

$$x \in \mathcal{Z} \quad \Longleftrightarrow \quad [x, y] = 0 \quad \forall\, y \in \mathfrak{A} \quad \Longrightarrow \quad [x, y, z] = 0 \quad \forall\, y, z \in \mathfrak{A}. \tag{2.18}$$

Since the associator is linear in its arguments, we can put (2.15), (2.11), and (2.18) together:

$$[x^{-1}, x, y] = \frac{[x^*, x, y]}{|x|^2} = \frac{2\langle 1, x\rangle [1, x, y] - [x, x, y]}{|x|^2} = 0 \qquad \forall\, x, y \in \mathfrak{A}, |x|^2 \neq 0. \tag{2.19}$$

Finally, we observe more general consequences of (2.11) and (2.18):

$$[x^*, y] = -[x, y] = [x, y]^* \qquad \forall\, x, y \in \mathfrak{A} \tag{2.20}$$

and

$$[x^*, y, z] = -[x, y, z] = [x, y, z]^* \qquad \forall\, x, y, z \in \mathfrak{A}, \tag{2.21}$$

which imply that both commutators and associators have vanishing inner products with 1:

$$\langle 1, [x, y]\rangle = \langle 1, [x, y, z]\rangle = 0 \qquad \forall\, x, y, z \in \mathfrak{A}. \tag{2.22}$$

We will now turn to the specific composition algebra of the octonions.

2.2.2. Octonions

According to a theorem by Hurwitz [16], which relies heavily on (2.13) there are only four composition algebras over the reals with a positive definite bilinear form, namely the reals, $\mathbb{R}$; the complexes, $\mathbb{C}$; the quaternions, $\mathbb{H}$ [17]; and the octonions or Cayley numbers, $\mathbb{O}$ [18]. Their dimensions as vector spaces over $\mathbb{R}$ are 1, 2, 4, and 8. Since the norm is positive definite, there exist inverses for all elements except 0 in these algebras. Therefore, they are also called normed division algebras.

For specific calculations the following concrete form of $\mathbb{O}$ is useful. $\mathbb{O} \cong \mathbb{R}^8$ as a normed vector space. Fortunately, it is always possible to choose an orthonormal basis $\{i_0, i_1, \ldots, i_7\}$ which induces a particularly simple multiplication table for the basis elements such as the one given by the following triples:

$$
\begin{gathered}
i_0 = 1, \\
i_a^2 = -1 \qquad (1 \leq a \leq 7), \\
i_a i_b = i_c = -i_b i_a \text{ and cyclic for} \\
(a,b,c) \in P = \{(1,2,3),(1,4,5),(1,6,7),(2,6,4),(2,5,7),(3,4,7),(3,5,6)\}.
\end{gathered}
\tag{2.23}
$$

The algorithm to obtain such a basis is similar to the Gram-Schmidt procedure [19] with additional requirements about products of the basis elements (see [4]).

Working over the field of real numbers, the following definitions of real and imaginary parts are customary:

$$
\begin{aligned}
\operatorname{Re} x &:= \langle 1, x\rangle = \tfrac{1}{2}(x + x^*) \in \mathbb{R}, \\
\operatorname{Im} x &:= x - \langle 1, x\rangle = \tfrac{1}{2}(x - x^*) \in \mathbb{R}^{\perp}.
\end{aligned}
\tag{2.24}
$$

Also i_0 is called the real unit and the other basis elements are called imaginary units,

$$
\operatorname{Re} i_0 = i_0, \qquad \operatorname{Im} i_a = i_a \qquad (1 \leq a \leq 7). \tag{2.25}
$$

In analogy to $\mathbb{C}$ and $\mathbb{H}$, the antiautomorphism $\cdot^*$ is called "octonionic conjugation". it also changes the sign of the imaginary part. With these conventions (2.22) reads

$$\operatorname{Re}[x,y] = \operatorname{Re}[x,y,z] = 0 \qquad \forall\, x,y,z \in \mathfrak{A}. \tag{2.26}$$

2.2.3. Multiplication tables

The question of possible multiplication tables arises, for example, when one reads another article on octonions, which, of course, uses a different one from the one given in (2.23). Usually it is remarked, that all 480 possible ones are equivalent, i.e., given an octonionic algebra with a multiplication table and any other valid multiplication table one can choose a basis such that the multiplication follows the new table in this basis. One may also take the point of view, that there exist different octonionic algebras, i.e., octonionic algebras with different multiplication tables. With this interpretation the previous statement means that all these octonionic algebras are isomorphic. However, this fact does not imply that a physical theory might not make use of more than one multiplication table at any given time. For our application, it will turn out that the limited symmetry of the physical theory leaves two classes of multiplication tables distinct.

We follow and expand the main ideas of Coxeter [20]. The set P in (2.23) can be taken to represent a labeling of the projective plane $\mathbb{Z}_2P^2$ over the field with two elements $\mathbb{Z}_2 = GF(2) = \{0,1\}$ (see Fig. 2.1). Before we explain this correspondence, we introduce the basic properties of $\mathbb{Z}_2P^2$. (Readers who are not familiar with projective geometry may consult [21].) This plane contains as points the one-dimensional linear subspaces of $(\mathbb{Z}_2)^3$. Given a basis of $(\mathbb{Z}_2)^3$ these subspaces are

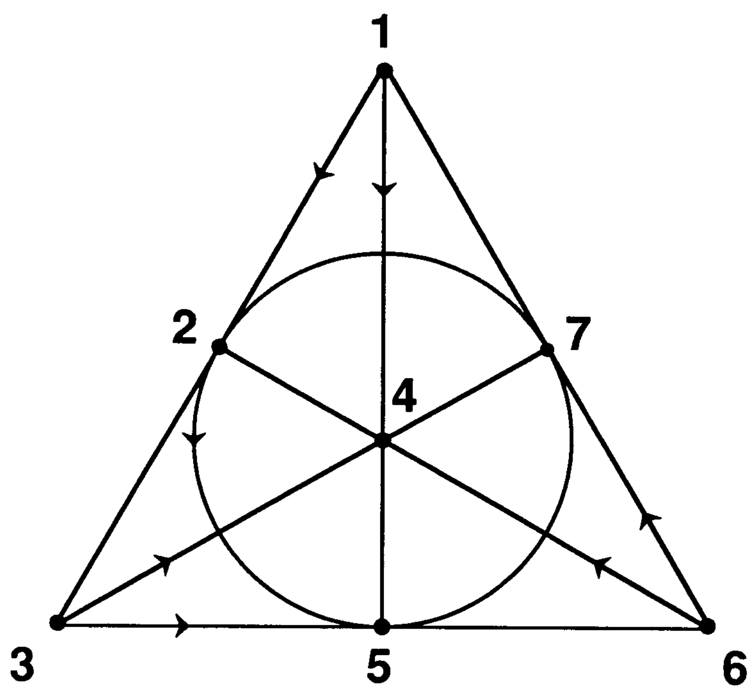

FIG. 2.1. The projective plane $\mathbb{Z}_2P^2$ representing a multiplication table for the octonions.

$$p_1 = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle, p_2 = \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle, p_3 = \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle, p_4 = \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle,$$
$$p_5 = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle, p_6 = \left\langle \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\rangle, p_7 = \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle. \tag{2.27}$$

(Since these linear subspaces contain only one non-zero element, we will drop the angle brackets and identify the points with the non-zero elements of $(\mathbb{Z}_2)^3$.) The lines l_1, l_2, $\ldots$, l_7 of the plane are the two-dimensional linear subspaces of $(\mathbb{Z}_2)^3$, which can also be described by their normal vectors n_1, n_2, $\ldots$, n_7, i.e., the dual vectors that annihilate the subspaces:

$$n_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, n_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, n_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, n_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$
$$n_5 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, n_6 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, n_7 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \tag{2.28}$$

So there are also seven lines in $\mathbb{Z}_2P^2$. The geometry of the plane is then defined by the incidence of points and lines, where

$$p_j \text{ and } l_k \text{ are incident} \quad \Longleftrightarrow \quad p_j \subset l_k \quad \Longleftrightarrow \quad n_k^T p_j \equiv 0 \pmod 2, \tag{2.29}$$

for example, p_3, p_5, and p_6 are incident with l_7.

We are now in a position to specify the previously mentioned correspondence between $\mathbb{Z}_2P^2$ and P. P contains seven triples formed out of seven labels. The labels represent points and the triples represent lines containing the three points given by the labels, i.e., a label and a triple are incident, if and only if the label is part of the triple. Cyclic permutations of a triple change neither the multiplication table nor the geometry of the plane. However, P does define an orientation on each line, since a transposition in a triple would change the multiplication table. This notion of orientation on the lines, is represented by arrows in Fig. 2.1. So we can read the multiplication table off the triangle. If we follow a line connecting two labels in direction of the arrow we obtain the product, for example, $i_3 i_4 = i_7$. When moving opposite to the direction of the arrow we pick up a minus sign, $i_4 i_2 = -i_6$. (Note that in projective geometry the ends of the lines are connected, i.e., lines are topologically circles, S^1.

What are possible transformations of the multiplication table P and how do they correspond to transformations of the projective plane $\mathbb{Z}_2 P^2$? Looking at Fig. 2.1, we see that there are three ways to change the picture:

(i) We may relabel the corners, leaving the arrows unchanged.

(ii) The labels may be kept fixed while some or all arrows are reversed.

(iii) Minus signs may be attached to the labels, i.e., we change part of (2.23) to read $i_a i_b = -i_c = -i_b i_a$ and cyclic for $(a, b, c) \in P$.

The sign change of a label in type (iii) is equivalent to reversing the orientation of the three lines through that point and therefore is included in the transformations of type (ii). For the second kind of transformation, we have to make sure, that the multiplication table so obtained satisfies alternativity, for it to define another octonionic algebra. One can show that given the arbitrary orientation of four lines including all seven points, the orientations of the remaining three lines are determined by alternativity. (Note that there is only one case to consider. Among the four lines there are necessarily three which have one point in common. Two of those together with the fourth one fix one of the remaining orientations.) This in turn implies that elementary transformations of type (ii) change the orientation of three lines which have one point in common. So the transformations of type (ii) and type (iii) are equivalent. Since four arrows can be chosen freely, we obtain sixteen as the number of possible configurations of arrows, i.e., the number of distinct multiplication tables that can be reached this way, namely: 1 original configuration with no changes, 7 with the orientation of three lines through one point reversed, 7 with the orientation of four lines avoiding one point reversed, and 1 with the orientation of all lines reversed.

In order to discuss these transformations further, we will introduce some notation. (Before developing this framework, I verified most of these results using the computer algebra package Maple. So the reader who is not algebraically inclined may take this proof by exhaustion as sufficient. For a basic reference on group theory see [22].) We denote an octonionic algebra given by an orthonormal basis of $\mathbb{R}^8$ and a set P of the type given in (2.23) by $\mathbb{O}_P$, and the set made up of all such octonionic algebras by $\mathcal{O} := \{$ all possible $P : \mathbb{O}_P\}$. "All possible P" means those that induce a multiplication table satisfying alternativity. So $\mathcal{O}$ can be viewed as the set of possible multiplication tables.

We now consider the group action of $T = T_1 * T_2$, the free product of transformations of type (i) and (ii), on $\mathcal{O}$:

$$\begin{aligned} T \times \mathcal{O} &\to \mathcal{O} \\ (t, \mathbb{O}_P) &\mapsto \mathbb{O}_{t(P)} \,. \end{aligned} \tag{2.30}$$

Thus each $t \in T$ induces an isomorphism $\mathbb{O}_P \xrightarrow{t} \mathbb{O}_{t(P)}$. The group of transformations T_1 of type (i), i.e., the relabelings of the corners, is of course the permutation group on seven letters, Σ_7, acting in the obvious way. We identify the group T_2 of transformations of type (ii) as $(\mathbb{Z}_2)^7$, with the 7 generators acting as the elementary transformations reversing the orientation of the three lines through one point. Earlier we saw that the orbits of an element of $\mathcal{O}$ under the action of this group are of size 16: $|\mathrm{Orb}_{(\mathbb{Z}_2)^7}(\mathbb{O}_P)| = 16$. In order to determine the orbits of Σ_7 we first consider its subgroup H which acts as the group of projective linear transformations on $\mathbb{Z}_2P^2$ labeled as in Fig. 2.1, i.e., we let H act on one specific $\mathbb{O}_P \in \mathcal{O}$, namely with P as in (2.23). $H \cong PGL(3,\mathbb{Z}_2) \cong GL(3,\mathbb{Z}_2)$ is generated by the permutations $(1\,2\,4\,3\,6\,7\,5)$ and $(1\,2\,5)(3\,7\,4)$. H is in fact simple, of Lie-type, of order $168 = 2^3 \cdot 3 \cdot 7$, and denoted by $A_2(2)$ (see [23]). Since elements of H as projective linear transformation do not change the geometry of $\mathbb{Z}_2P^2$, they can only reverse the orientations of lines, i.e., $\mathrm{Orb}_H(\mathbb{O}_P) \subseteq \mathrm{Orb}_{(\mathbb{Z}_2)^7}(\mathbb{O}_P)$. Hence, we have $|\mathrm{Orb}_{H*(\mathbb{Z}_2)^7}(\mathbb{O}_P)| = 16$. Thus the index of the stabilizing subgroup of H has to divide 16:

$$[H : \mathrm{Stab}_H(\mathbb{O}_P)] = |\mathrm{Orb}_H(\mathbb{O}_P)| \,\Big|\, 16. \tag{2.31}$$

Since the action of H is not trivial and H being simple of order 168 cannot have subgroups of index 2 or 4, we conclude $|\mathrm{Orb}_H(\mathbb{O}_P)| = 8$. To determine $|\mathrm{Orb}_{\Sigma_7}(\mathbb{O}_P)|$ we need to consider the cosets of H in Σ_7. There are $[\Sigma_7 : H] = 30$ of them corresponding to distinct geometries of $\mathbb{Z}_2P^2$, i.e., the incidence of lines and points is different for different cosets. Therefore, there are 30 distinct classes of multiplication tables, with members of one class related by a projective linear transformation. So it follows

$$\begin{aligned} |\mathrm{Orb}_{\Sigma_7}(\mathbb{O}_P)| &= 30 \cdot 8 = 240, \\ |\mathrm{Orb}_T(\mathbb{O}_P)| &= 30 \cdot 16 = 480. \end{aligned} \tag{2.32}$$

So relabelings of the corners reach only half of the possible multiplication tables, which is a consequence of the fact that projective linear transformations reach only half of the possible configurations of arrows. Why is this so and what are the possible implications? To answer these questions we need to understand how elements of H

change orientations of lines. We can decompose the action of elements of H into one part that permutes the lines and another one that reverses the orientation of certain lines in the image. An element $t_1 \in H$ of odd order p may only change the orientation of an even number of lines. For $t_1{}^p = 1$ has to act trivially on P, and the changes of orientation add up modulo 2. However, H is generated by elements of odd order, so all of its elements change only the orientation of an even number of lines. To obtain the full orbit we may add just one element $\zeta \in T_2$ that changes the orientation of an odd number of lines. A particularly good choice for ζ is the product of all generators, i.e., the one corresponding to reversing all seven lines (or attaching minus signs to all labels when viewed as type (iii) transformation). Obviously, $t_1\,\zeta(P) = \zeta\,t_1(P) \quad \forall t_1 \in T_1$, so that we may form the direct product $T_1{}' = T_1 \times \{1, \zeta\}$ and $\mathrm{Orb}_{T_1{}'}(\mathbb{O}_P) = \mathrm{Orb}_T(\mathbb{O}_P)$. Note that ζ corresponds to the operation of octonionic conjugation, so that the isomorphism given by ζ is illustrated by the following diagram:

$$\begin{array}{ccc}
\mathbb{O}_P \times \mathbb{O}_P & \longrightarrow & \mathbb{O}_P \\
(a, b) & \longmapsto & ab \\
\zeta \times \zeta \big\downarrow & \circlearrowleft & \big\downarrow \zeta \\
\mathbb{O}_{\zeta(P)} \times \mathbb{O}_{\zeta(P)} & \longrightarrow & \mathbb{O}_{\zeta(P)} \\
(a^*, b^*) = (a', b') & \longmapsto & (ab)^* = b^*\,a^* = b'\,a'
\end{array} \quad . \tag{2.33}$$

Therefore, $\mathbb{O}_{\zeta(P)}$ is the opposite algebra of $\mathbb{O}_P$, i.e., the algebra obtained by reversing the order of all products. So for octonionic algebras, there is an isomorphism of an algebra and its opposite algebra given by octonionic conjugation, besides the natural anti-isomorphism given by identification. What are the consequences of these results for a physical theory? Usually, the physical theory will contain a vector space of dimension 8, for which we want to introduce an octonionic description.

This description, however, should be invariant under the appropriate symmetry group, most commonly, $SO(8)$. The multiplication table changes in a more general way under $SO(8)$. The product of two basis elements will turn out to be a linear combination of all basis elements, but the relabelings given by Σ_7 are certainly a subgroup contained in $SO(8)$. Moreover, $\zeta \notin SO(8)$, which implies that the most general multiplication tables with respect to an orthonormal basis split in two classes with $SO(8)$ acting transitively on each class, but only $SO(8) \times \{1, \zeta\} \cong O(8)$ acting transitively on all of them. In fact we will find it useful to consider two algebra structures, namely $\mathbb{O}$ and its opposite $\mathbb{O}_{\mathrm{opp}}$, on the same $\mathbb{R}^8$ to describe the spinors of opposite chirality.

In a recent article, Cederwall & Preitschopf [24] introduce an "X-product" on $\mathbb{O}$ via

$$a \underset{X}{\circ} b := (a\,X)(X^*\,b), \qquad a, b, X \in \mathbb{O},\ X\,X^* = 1, \tag{2.34}$$

which is just the original product for $X = 1$. As X becomes different from 1, the multiplication table for this product changes continuously in a way related to the $SO(8)$ transformations that leave 1 fixed. This changing product appears naturally when the basis of a spinor space is changed, see section 2.4.5.

2.3. CLIFFORD ALGEBRAS AND THEIR REPRESENTATIONS

The second building block for octonionic representations of Clifford algebras is presented in this section. First we define an abstract Clifford algebra and observe some of its basic properties. Then we consider the Clifford group which gives us the action of the orthogonal groups on vectors and spinors. The third subsection states the necessary facts about representations of Clifford algebras, i.e., how we can find matrix algebras to describe Clifford algebras. For further reference and proofs that are left out see [25,9,26,27]. We only consider the real or complex field, i.e., $\mathbb{F} = \mathbb{R}, \mathbb{C}$ in this section, even though some of the statements generalize to other fields in particular of characteristic different from 2.

2.3.1. Clifford algebras

The tensor algebra $\mathcal{T}(V)$ of a vector space V of dimension n over a field $\mathbb{F}$ is the free associative algebra over V: (All the products in this section are associative.)

$$\mathcal{T}(V) := \oplus_{k=0}^{\infty}(V)^k, \tag{2.35}$$

where

$$(V)^n = \underbrace{V \otimes V \otimes \cdots \otimes V}_{n \text{ copies}}, \; n > 0, \quad (V)^0 = \mathbb{F}. \tag{2.36}$$

The identity element is $1 \in \mathbb{F}$ and $\mathbb{F}$ lies in the center of $\mathcal{T}(V)$. Given a metric g on V, i.e., g is a non-degenerate symmetric bilinear form on V, the Clifford algebra $\mathcal{C}l(V, g)$ is defined to be

$$\mathcal{C}l(V, g) := \mathcal{T}(V)/I(g)\,, \tag{2.37}$$

where

$$I(g) = \langle u \otimes u - g(u, u) : u \in V \rangle \tag{2.38}$$

is the two-sided ideal generated by all expressions of the form $u \otimes u - g(u, u)$. If V is unambiguously defined from the context, we simply write $\mathcal{C}l(g)$. We denote multiplication in $\mathcal{C}l(g)$ by

$$u \vee v := \pi^{-1}(u) \otimes \pi^{-1}(v) + I(g) \qquad \forall u, v \in \mathcal{C}l(g), \tag{2.39}$$

where π is the canonical projection:

$$\begin{aligned} \pi : \mathcal{T}(V) &\to Cl(g) \\ u &\mapsto u + I(g) \end{aligned} \tag{2.40}$$

and $\pi^{-1}(u)$ is any preimage of u. Since π restricted to $\mathbb{F} \oplus V$ is injective, we identify this space with its embedding in $Cl(g)$.

From a more practical perspective a Clifford product is just a tensor product with the additional rule that

$$u \vee u = g(u, u) \qquad \forall\, u \in V. \tag{2.41}$$

As a consequence elements of $V \subseteq Cl(g)$ anticommute up to an element of $\mathbb{F}$:

$$\{u, v\} := u \vee v \,+\, v \vee u = 2g(u, v) \qquad \forall\, u, v \in V \tag{2.42}$$

or in terms of an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$

$$\{e_i, e_j\} := 2g(e_i, e_j) = \begin{cases} \pm 2, & \text{for } i = j \\ 0, & \text{for } i \neq j \end{cases} \qquad (1 \leq i, j \leq n). \tag{2.43}$$

Based on these relationships, we find a basis for $Cl(g)$ as a vector space,

$$\{e_{a_1} \vee e_{a_2} \vee \cdots \vee e_{a_k} : 0 \leq k \leq n,\, 1 \leq a_1 < a_2 < \cdots \leq n\}, \tag{2.44}$$

which shows that

$$\dim Cl(g) = \sum_{k=0}^{n} \binom{n}{k} = 2^n. \tag{2.45}$$

The product $\eta = e_1 \vee e_2 \vee \cdots \vee e_n$ is called the volume form and has the special property

$$\eta \vee u = (-1)^{n+1} u \vee \eta \qquad \forall\, u \in V. \tag{2.46}$$

So for odd n, η lies in the center $\mathcal{Z}$ of $Cl(g)$. In fact

$$\mathcal{Z} = \begin{cases} \mathbb{F}, & \text{for } n \text{ even} \\ \mathbb{F} \oplus \mathbb{F}\eta, & \text{for } n \text{ odd} \end{cases} . \tag{2.47}$$

There are two involutions on $\mathcal{T}(V)$: the main automorphism α,

$$\begin{aligned} \alpha|_V : V &\to V \\ u &\mapsto -u\,, \end{aligned} \tag{2.48}$$

and the main antiautomorphism β,

$$\begin{aligned} \beta|_V : V &\to V \\ u &\mapsto u \end{aligned} \tag{2.49}$$

$$\beta(u \otimes v) = v \otimes u \quad \forall u, v \in V\,.$$

These restrictions can obviously be extended to an automorphism and an antiautomorphism of $\mathcal{T}(V)$. Since $I(g)$ is invariant under α and β, we obtain maps on the quotient $\mathcal{C}l(g)$. The main antiautomorphism can also be understood as an isomorphism of $\mathcal{C}l(g)$ and its opposite algebra $(\mathcal{C}l(g))_{\text{opp}}$:

$$\begin{array}{ccc} \mathcal{C}l(g) \times \mathcal{C}l(g) & \xrightarrow{\vee} & \mathcal{C}l(g) \\ (a,b) & \longmapsto & a \vee b \\ \beta \times \beta \big\downarrow & \circlearrowleft & \big\downarrow \beta \\ (\mathcal{C}l(g))_{\text{opp}} \times (\mathcal{C}l(g))_{\text{opp}} & \xrightarrow{\vee_{\text{opp}}} & (\mathcal{C}l(g))_{\text{opp}} \\ (\beta(a), \beta(b)) = (a_{\text{opp}}, b_{\text{opp}}) & \longmapsto & a_{\text{opp}} \vee_{\text{opp}} b_{\text{opp}} = b_{\text{opp}} \vee a_{\text{opp}} \\ & & = \beta(b) \vee \beta(a) = \beta(a \vee b) \end{array} . \tag{2.50}$$

We no longer have the $\mathbb{Z}$-grading of $\mathcal{T}(V)$ given by the rank, but the main automorphism α defines a $\mathbb{Z}_2$-grading on $\mathcal{C}l(g)$ given by the projections

$$P_0 := \frac{1}{2}(\text{id} + \alpha), \qquad P_1 := \frac{1}{2}(\text{id} - \alpha). \tag{2.51}$$

The even and odd part of the Clifford algebra are defined to be

$$\mathcal{C}l_0(g) := P_0(\mathcal{C}l(g)), \qquad \mathcal{C}l_1(g) := P_1(\mathcal{C}l(g)). \tag{2.52}$$

Then

$$\mathcal{C}l(g) = \mathcal{C}l_0(g) \oplus \mathcal{C}l_1(g) \tag{2.53}$$

and the even part $\mathcal{C}l_0(g)$ is in fact a subalgebra of $\mathcal{C}l(g)$.

We already saw how the Clifford algebra contains vectors. A spinor space S is defined to be a minimal left ideal of $\mathcal{C}l(g)$. Such a space

$$S = \mathcal{C}l(g) \vee Q \tag{2.54}$$

is generated by a primitive idempotent $Q \in \mathcal{C}l(g)$, i.e.,

$$Q^2 = Q, \qquad \nexists\, Q_1, Q_2 : \; Q_1^2 = Q_1 \neq 0, \; Q_2^2 = Q_2 \neq 0, \; Q = Q_1 + Q_2. \tag{2.55}$$

(This characterization of minimal left ideals relies on the fact that Clifford algebras over $\mathbb{R}$ and $\mathbb{C}$ are semisimple, see section 2.3.3.) If the primitive idempotent is even, the spinor space S decomposes into the spaces of even and odd Weyl spinors:

$$S = S_0 \oplus S_1, \qquad S_k = P_k S = \mathcal{C}l_k(g) \vee Q, \; (k = 0, 1). \tag{2.56}$$

There are different names for these spaces which are being used within the mathematical physics community. S is also called the space of Dirac spinors and S_0 and S_1 are called semi-spinor spaces. Sometimes, elements of S are called bi-spinors and elements of S_0 and S_1 are just called even and odd spinors. For mixed primitive idempotent Q there may still be a Weyl decomposition (2.86), but it is not compatible with the $\mathbb{Z}_2$ grading on $\mathcal{C}l(g)$:

$$S = \mathcal{C}l(g) \vee Q = \mathcal{C}l_0(g) \vee Q = \mathcal{C}l_1(g) \vee Q. \tag{2.57}$$

For odd n, S is also called the space of Pauli spinors or semi-spinors. If only the double $2S := S \oplus S$ carries a faithful representation of $Cl(g)$ (see section 2.3.3), then some authors refer only to $2S$ as the space of spinors.

2.3.2. The Clifford group

The connection of the symmetry group of the metric, i.e., the orthogonal group, with the Clifford algebra is made in this subsection via the Clifford group $\Gamma(g)$. We define the Clifford group $\Gamma(g)$ to be the group generated by the vectors of non-zero norm, i.e.,

$$\Gamma(g) := \langle u \in V \subseteq Cl(g) : u^2 = g(u,u) \neq 0 \rangle. \tag{2.58}$$

As we will see, this definition is almost equivalent to the usual one,

$$\Gamma'(g) := \{u \in Cl(g) : u \text{ invertible, } u \vee x \vee u^{-1} \in V \ \ \forall x \in V\} \supseteq \Gamma(g). \tag{2.59}$$

Considering $u \in \Gamma(g) \cap V$ and any $x \in V$ we see that

$$\begin{aligned} u \vee x \vee u^{-1} &= u \vee x \vee \tfrac{1}{g(u,u)} u = \tfrac{1}{g(u,u)}(-x \vee u + 2g(x,u)) \vee u \\ &= -x + 2\tfrac{g(x,u)}{g(u,u)} u \in V. \end{aligned} \tag{2.60}$$

Therefore, $\Gamma'(g) \supseteq \Gamma(g)$ indeed, and in particular $\Gamma'(g) \cap V = \Gamma(g) \cap V$. In fact, the definition of $\Gamma'(g)$ implies that $\Gamma(g)$ is stable under conjugation in $\Gamma'(g)$, i.e., $\Gamma(g)$ is a normal subgroup of $\Gamma'(g)$. We will investigate the structure of the Clifford group on the basis of this group action of $\Gamma'(g)$ on V:

$$\begin{aligned} \phi' : \Gamma'(g) \times V &\to V \\ (u, x) &\mapsto \phi'_u(x) := u \vee x \vee u^{-1}. \end{aligned} \tag{2.61}$$

Dropping all the primes we have the obvious restriction

$$\begin{aligned} \phi : \Gamma(g) \times V &\to V \\ (u, x) &\mapsto \phi_u(x) := u \vee x \vee u^{-1}. \end{aligned} \tag{2.62}$$

(We will not explicitly give the unprimed analogues of expressions below.) Of course, these actions can be extended to give inner automorphisms of $Cl(g)$. According to (2.60), the action of $u \in V \cap \Gamma'(g)$ is just a reflection of x at the hyperplane orthogonal to u composed with an inversion of the whole space. In particular $\phi'_u(x) \in V$ and ϕ'_u is an isometry:

$$\begin{aligned} g(\phi'_u(x), \phi'_u(x)) = \phi'_u(x) \vee \phi'_u(x) &= (\tfrac{1}{u^2})^2 u \vee x \vee u^{-1} \vee u \vee x \vee u^{-1} \\ &= u \vee x \vee x \vee u^{-1} = g(x,x). \end{aligned} \tag{2.63}$$

So (2.61) (resp. (2.62)) gives a homomorphism Φ' (resp. Φ) of $\Gamma'(g)$ (resp. $\Gamma(g)$) to the group of isometries or orthogonal transformations $O(g)$ of V:

$$\begin{aligned} \Phi' : \Gamma'(g) &\to O(g) \\ u \mapsto \phi'_u &: V \to V \\ & \qquad x \mapsto \phi'_u(x) = u \vee x \vee u^{-1} \end{aligned} \tag{2.64}$$

To compare $\Gamma'(g)$ (resp. $\Gamma(g)$) with $O(g)$ we need to know the range and the kernel of Φ' (resp. Φ). Since the reflections at hyperplanes generate all orthogonal transformations Φ' (resp. Φ) is onto, if we can find a preimage of the inversion $x \mapsto -x$. Because of (2.46), $\eta \in \Gamma(g) \subseteq \Gamma'(g)$ does the job for even n. For odd n, there is no element of $Cl(g)$ that anticommutes with all $x \in V$. So there is no preimage of the inversion , which leaves us with $SO(g)$ as the range. The kernel coincides with the part of the center, that lies in the Clifford group. Thus we have according to the homomorphism theorems

$$\Gamma(g)/_{\mathbb{F}^*} \cong O(g) \cong \Gamma'(g)/_{\mathbb{F}^*} \qquad \text{(for even } n\text{)} \tag{2.65}$$

$$\Gamma(g)/_{\mathbb{F}^*\langle\eta\rangle} \cong SO(g) \cong \Gamma'(g)/_{\mathcal{Z}^*} \qquad \text{(for odd } n\text{),} \tag{2.66}$$

where $\mathbb{F}^* = \mathbb{F} \setminus \{0\}$, $\langle\eta\rangle$ is the group generated by η, and $\mathcal{Z}^* := \Gamma'(g) \cap \mathcal{Z}$ is the invertible part of the center. So the Clifford group is isomorphic to the orthogonal (resp. simple orthogonal) group up to a subgroup of the center $\mathcal{Z}$. Therefore,

$$\Gamma(g) \cong \Gamma'(g) \qquad \text{(for even } n\text{)} \tag{2.67}$$

$$\Gamma(g) \times {}^{\mathcal{Z}^*}\!/_{\mathbb{F}^*\langle\eta\rangle} \cong \Gamma'(g) \qquad \text{(for odd } n\text{)}, \tag{2.68}$$

So for even n both definitions (2.58) and (2.59) of the Clifford group are equivalent. For odd n they differ by inhomogeneous elements of the invertible part of the center $\mathcal{Z}^*$. For our purposes it will be sufficient to consider the Clifford group $\Gamma(g)$ as defined in (2.58) only.

For both even and odd n, we obtain a homomorphism from the even Clifford group $\Gamma_0(g)$,

$$\Gamma_0(g) := \Gamma(g) \cap \mathcal{C}l_0(g) = P_0\Gamma(g) = \Gamma'(g) \cap \mathcal{C}l_0(g) = P_0\Gamma'(g), \tag{2.69}$$

onto $SO(g)$:

$$\Gamma_0(g)/_{\mathbb{F}^*} \cong SO(g). \tag{2.70}$$

The even Clifford group is generated by pairs of vectors with non-zero norm:

$$\Gamma_0(g) := \langle u \vee v : u, v \in V,\ g(u,u) \neq 0 \neq g(v,v)\rangle. \tag{2.71}$$

In fact one of the vectors may be fixed,

$$\Gamma_0(g) := \langle u \vee w : u \in V,\ g(u,u) \neq 0\rangle, \quad \text{for some } w \in V,\ g(w,w) \neq 0, \tag{2.72}$$

since inverses are of the same form: $(u_\vee w)^{-1} = w^{-1}{}_\vee u^{-1} = (w^{-1}{}_\vee u^{-1}{}_\vee w^{-1})_\vee w = v_\vee w$, for some $v \in V$.

We also have an action ψ of $\Gamma(g)$ on the Clifford algebra $\mathcal{C}l(g)$ and in particular on any of its minimal left ideals, a space of spinors S:

$$\psi : \Gamma(g) \times S \to S \tag{2.73}$$

$$(u, s) \mapsto \psi_u(s) := u \vee s\,.$$

2.3.3. Representations of Clifford algebras

In this subsection we describe how we can get a matrix algebra that is isomorphic to a Clifford algebra. In a sense this is the analogue to 2.2.2, where we gave an explicit form of the octonions, which implemented their abstract properties. We start out introducing some definitions concerning representations in general. Algebras are assumed to be finite dimensional and contain a unit element. (For a general reference for representation theory see [28].)

A representation γ of an algebra $\mathfrak{A}$ over a field $\mathbb{F}$ in a vector space W is a homomorphism

$$\begin{aligned} \gamma : \mathfrak{A} &\to \mathrm{End}_{\mathbb{F}}(W) \\ a &\mapsto \gamma(a) : W \to W \\ &\qquad\qquad w \mapsto \gamma(a)w\,, \end{aligned} \tag{2.74}$$

i.e.,

$$\left.\begin{aligned} \gamma(a \vee b) &= \gamma(a)\gamma(b) \\ \gamma(a+b) &= \gamma(a)+\gamma(b) \end{aligned}\right\} \qquad \forall\, a, b \in \mathfrak{A}. \tag{2.75}$$

Given a basis of W, $\gamma(a)$ as an endomorphism of W may be understood as an $l \times l$-matrix, where $l = \dim W$ is called the dimension of the representation. The representation is called faithful, if γ is injective. R is an invariant subspace of γ, if $\gamma(a)R \subseteq R \quad \forall a \in \mathfrak{A}$. The representation γ is called irreducible, if there are no invariant subspaces of γ other than $W \neq \{0\}$ and $\{0\}$. A reducible representation γ may be reduced to a representation γ_R on an invariant subspace R, i.e., $\mathfrak{A} \xrightarrow{\gamma_R} \mathrm{End}_{\mathbb{F}}(R)$ requiring $\gamma_R(a)w = \gamma(a)w \quad \forall w \in R,\ a \in \mathfrak{A}$. An algebra is called simple, if it allows a faithful and irreducible representation. An algebra is called semisimple if it is a direct sum of simple algebras.

Since a left ideal J of $\mathfrak{A}$ is by definition stable under left multiplication,

$$\mathfrak{A}_{\vee} J \subseteq J, \tag{2.76}$$

and is a vector space, we have a natural representation λ_J of $\mathfrak{A}$ on J. (Again given a basis $\{b_1, b_2, \ldots, b_l\}$ we have a representation in terms of matrices: $a_\vee b_i = \lambda_J(a)_{ij} b_j$.) Taking $J = \mathfrak{A}$ we obtain the so called left regular representation, which is faithful. If J is a minimal left ideal, then the representation on it is irreducible, since invariant subspaces would correspond to proper subspaces of J which are left ideals and contradict the minimality of J.

If the algebra $\mathfrak{A}$ is semisimple then the converse is also true, i.e., any irreducible representation can be written as a λ_J for some minimal left ideal J: In this case an irreducible representation γ of $\mathfrak{A} = \mathfrak{A}_1 \oplus \mathfrak{A}_2 \oplus \cdots \oplus \mathfrak{A}_k$ is an irreducible representation of one of the simple components, say $\mathfrak{A}_k$. So a minimal ideal L of $\gamma(\mathfrak{A})$ can be lifted to a minimal ideal of $J \subseteq \mathfrak{A}_j$, such that $\gamma(J) = L$. Then the following diagram commutes,

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{\gamma} & \gamma(\mathfrak{A}) \subseteq \mathrm{End}(W) \\ \lambda_J \big\downarrow & \circlearrowleft & \big\downarrow \lambda_L \\ \mathrm{End}(J) & \xrightarrow{\gamma(J)=L} & \mathrm{End}(L) \end{array} \quad . \tag{2.77}$$

Since the maps λ_L and $\gamma(J) = L$ are isomorphisms, there is an isomorphism relating W and J as vector spaces,

$$F : W \to J, \tag{2.78}$$

such that

$$\gamma(a) \circ F = F \circ \lambda_J(a) \qquad \forall\, a \in \mathfrak{A}. \tag{2.79}$$

F is said to intertwine the representations γ and λ_J:

$$\begin{array}{ccc} W & \xrightarrow{\gamma(a)} & W \\ F\big\downarrow & \circlearrowleft & \big\downarrow F \\ J & \xrightarrow{\lambda_J} & J \end{array} \quad . \tag{2.80}$$

Representations related in this way are called equivalent. In terms of their matrix form, equivalent representations are related by a basis transformation. This observation also shows that for a simple algebra all irreducible representations are equivalent to λ_J and therefore equivalent to each other.

As it is shown in the references given (see in particular [25,27]), Clifford algebras over $\mathbb{R}$ and $\mathbb{C}$ are simple or semisimple. Therefore, there is an equivalent definition for spinors in terms of representations of $Cl(g)$, i.e., a spinor space S can be defined to be the carrier space of an irreducible representation of $Cl(g)$.

In order to find a concrete form of a representation, we are still left with the task of finding a primitive idempotent Q that generates a minimal left ideal J and observing how the basis elements of $Cl(g)$ act on it. Actually, we will give a procedure to construct a representation that does not use a primitive idempotent explicitly. For this purpose we define the signature of a metric for the case $\mathbb{F} = \mathbb{R}$. We say that g has the signature p, q (write $g_{p,q}$), $\dim V = p + q = n$, if there is a basis $\{e_1, e_2, \ldots, e_n\}$ of V, such that

$$g_{ij} := g(e_i, e_j) = \begin{cases} 0, & \text{for } i \neq j \\ 1, & \text{for } i = j \leq p \\ -1, & \text{for } i = j > p \end{cases} . \tag{2.81}$$

(For $\mathbb{F} = \mathbb{C}$, given an orthonormal basis the transformation $e_k \mapsto \left\{ \begin{smallmatrix} e_k, & \text{for } k \neq j \\ ie_j, & \text{for } k=j \end{smallmatrix} \right.$ changes the sign of $g(e_j, e_j)$, i.e., we may choose a basis to obtain a form (2.81) of the metric with any p, q where $p + q = n$. Therefore, the procedure given applies to the complex case also.) We write $Cl(p,q)$ and $\gamma_{p,q}$ to denote $Cl(g_{p,q})$ and one of its representations. It is particularly simple to give a procedure that produces a representation of $Cl(m,m)$, i.e., in the case of a so-called neutral space. The procedure starts by "guessing" a representation $\gamma_{1,1}$ for $Cl(1,1)$:

$$\begin{aligned} \gamma_{1,1}(e_1) &:= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} =: \sigma \quad \text{and} \quad \gamma_{1,1}(e_2) := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} =: \epsilon \\ \Longrightarrow \quad \gamma_{1,1}(e_1 \vee e_2) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} =: \tau \quad \text{and} \quad \gamma_{1,1}(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{1}. \end{aligned} \tag{2.82}$$

Notice that the representation is completely specified by defining it on a basis of V, since V generates the algebra. In order to ensure that these assignments actually lead to a representation of the Clifford algebra, we need to check that (2.42) is satisfied for all pairs of images of basis elements, i.e.,

$$\{\gamma(e_i), \gamma(e_j)\} = \gamma(e_i)\gamma(e_j) + \gamma(e_j)\gamma(e_i) = 2g_{ij} \qquad (1 \leq i, j \leq n). \tag{2.83}$$

The representation $\gamma_{1,1}$ is faithful and irreducible, since its image is the space $M_2(\mathbb{F})$ of 2×2-matrices. So there are no proper invariant subspaces and the dimensions of $Cl(1,1)$ and $M_2(\mathbb{F})$ match. This representation may be used as a building block to extend a faithful irreducible representation $\gamma_{p,q}$ of $Cl(p,q)$, ($p+q=2m$ even) to a representation γ' of $Cl(V', g')$ with $\dim V' = 2m+2$:

$$\begin{aligned} \gamma'(e'_i) &= \sigma \otimes \gamma_{p,q}(e_i) \qquad (1 \leq i \leq 2m), \\ \gamma'(e'_{2m+1}) &= \sigma \otimes \gamma_{p,q}(\eta), \qquad \gamma'(e'_{2m+2}) = \epsilon \otimes \gamma_{p,q}(1). \end{aligned} \tag{2.84}$$

(Of course, there are possible extensions using the same building blocks, other than this so-called Cartan extension.) It is easy to check that γ' is faithful and irreducible if $\gamma_{p,q}$ was. The signature of the resulting metric g' depends on the value of

$$\gamma_{p,q}(\eta)^2 = (-1)^{\frac{1}{2}\nu(\nu-1)}\,\mathbf{1}, \tag{2.85}$$

where $\nu := p - q = 2(p - m) = 2(m - q)$ is called the index of the metric $g_{p,q}$. So for even (resp. odd) $\frac{\nu}{2}$, we obtain a representation γ' of $Cl(p+1, q+1)$ (resp. $Cl(p, q+2)$). Since for neutral spaces $\nu = 0$, we can get any $\gamma_{m,m}$, starting from $\gamma_{1,1}$ by iteration of this extension. We note that the dimension l of the carrier space of this irreducible representation is $2^m = 2^{\frac{n}{2}}$.

For even $\frac{\nu}{2}$, $\gamma_{p,q}(\eta)$ has eigenvalues $+1$ and -1 and we have Weyl projections $P_\pm$ (2.56):

$$P_\pm := \frac{1}{2}(1 \pm \eta). \tag{2.86}$$

One of these projectors can be decomposed to give an even primitive idempotent Q. A representation such as the one given, where $\gamma_{p,q}(\eta) = \begin{pmatrix} 1_{m\times m} & 0 \\ 0 & -1_{m\times m} \end{pmatrix}$ is called a Weyl representation, since the Weyl projections $P_\pm$ take a simple form. Due to the property (2.46) of η,

$$P_\pm a = a_0 P_\pm + a_1 P_\mp, \tag{2.87}$$

where a_0 and a_1 are the even and odd part of a. Since either P_+ or P_- annihilates the even primitive idempotent Q, we indeed get projections onto the spaces of even and odd Weyl spinors. Let, for example, $P_+ \vee Q = Q$ and $P_- \vee Q = 0$, then for $s = a \vee Q = a_0 \vee Q + a_1 \vee Q \in S_0 \oplus S_1$, $a = a_0 + a_1$ as before,

$$\begin{aligned} P_+ \vee s &= P_+ \vee a \vee Q = a_0 \vee P_+ \vee Q + a_1 \vee P_- \vee Q = a_0 \vee Q \in S_0, \\ P_- \vee s &= P_- \vee a \vee Q = a_0 \vee P_- \vee Q + a_1 \vee P_+ \vee Q = a_1 \vee Q \in S_1. \end{aligned} \tag{2.88}$$

If we choose a mixed primitive idempotent Q, then we get a different decomposition $S = P_+ S \oplus P_- S$ unrelated to the decomposition of the Clifford algebra in its even and odd part.

We now construct representations for even n and $\nu \neq 0$. In this case we can get a complex representation of the same dimension $l = 2^m$ by complexifying and transforming the metric to obtain a neutral space. This complex representation is faithful and irreducible but not necessarily equivalent to a real one. To examine this issue we define the complex conjugate γ^* of a representation $\gamma : \mathfrak{A} \longrightarrow \mathrm{End}_{\mathbb{C}}(W)$ by

$$\begin{aligned} \gamma^* : \mathfrak{A} &\longrightarrow \mathrm{End}_{\mathbb{C}}(W^*) \\ a &\mapsto \gamma^*(a) = (\gamma(a))^* . \end{aligned} \tag{2.89}$$

If $\mathfrak{A}$ is simple then γ and γ^* are equivalent, i.e., there exists a linear map $C : W \longrightarrow W^*$ intertwining these two representations:

$$\gamma^*(a) \circ C = C \circ \gamma(a) \qquad \forall\, a \in \mathfrak{A}. \tag{2.90}$$

It follows by complex conjugation that

$$\gamma(a) \circ C^* \circ C = C^* \circ \gamma^*(a) \circ C = C^* \circ C \circ \gamma(a) \qquad \forall\, a \in \mathfrak{A}, \tag{2.91}$$

whence by Schur's Lemma $C^* \circ C$ is proportional to the identity. Since $C^* \circ C$ has a real eigenvalue, C can be normalized to satisfy

$$C^* \circ C = \pm \mathbf{1}. \tag{2.92}$$

If and only if $C^* \circ C = +\mathbf{1}$, then we can find a basis transformation to make $\gamma_{p,q}$ real. This is the case for $\nu \equiv 0, 2 \pmod 8$. In practice, we relate W and W^* by complex conjugation in the obvious way. C is found by imposing (2.90) for $a \in \{e_1, e_2, \ldots, e_n\}$. (Following the procedure given above, any of the matrices $\gamma(e_k)$ is either real or purely imaginary, so that C either commutes or anticommutes with it.) The new basis is a basis of eigenvectors for C, which is invariant under $s \mapsto s_C := (C\, s)^*$. ($s_C$ is essentially the charge conjugate spinor for s.) For the cases $\nu \equiv 0, 6 \pmod 8$ we can make a similar transformation to make $\gamma_{p,q}$ purely imaginary. These real (resp. purely imaginary) representations are known as Majorana representations of the first (resp. second) kind. Of course, even for $\nu \equiv 4, 6 \pmod 8$ we can find an irreducible real representation of higher dimension, namely $l = 2^{m+1}$, by letting $1 \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{1}$ and $i \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \epsilon$ in an irreducible complex representation.

From a faithful, irreducible representation γ of the full Clifford algebra $Cl(g)$ we derive a representation γ_0 of the even subalgebra $Cl_0(g)$ by the obvious restriction. γ_0 is faithful, but not irreducible, except for real representations when $\nu \equiv 2 \pmod 8$. For $\nu \equiv 0, 4 \pmod 8$, there are two-sided ideals of $Cl_0(g)$ generated by the idempotents $\frac{1}{2}(1 \pm \eta)$. Each of these two-sided ideals J carries an irreducible representation of dimension 2^{m-1}, but only the double $2J$ carries a faithful representation. For $\nu \equiv 6 \pmod 8$ the isomorphism (2.93) in the following paragraph shows that $Cl_0(q,p) \cong Cl_0(p,q)$, hence we know the dimension of the irreducible representation to be $l = 2^m$ from the case $\nu \equiv 2 \pmod 8$.

Representations $\gamma_{p,q}$ with odd n can be obtained by shrinking a representation of higher dimension, since we have the isomorphisms

$$Cl_0(q+1,p) \cong Cl(p,q) \cong Cl_0(p,q+1) \tag{2.93}$$

obtained from extending

$$e_1 e_{k+1} \leftarrow\!\shortmid e_k \mapsto e_k e_{n+1} \qquad (1 \le k \le n). \tag{2.94}$$

Given the procedure above we can find an irreducible representation of $Cl(p,q)$ by constructing one corresponding to an even subalgebra for even n. According to the isomorphism (2.93) which also holds true for $p+q$ even, we can shrink representations for odd n in a similar way.

Irreducible representations of the Clifford algebra $Cl(g)$ induce irreducible representations of the Clifford group $\Gamma(g)$, since the basis elements of $Cl(g)$ as in (2.44) are contained in $\Gamma(g)$. The representations arising from the tensor (resp. spinor) action (2.62) (resp. (2.73)) are known as the vector (resp. spinor) representation of $\Gamma(g)$.

2.3.4. Bilinear forms on spinors

Physical observables are tensors, which in terms of the Clifford algebra transform like (2.62), while spinors transform like (2.73) under the orthogonal group. For this reason it seems likely that a bilinear form on spinors may provide observables based on spinors. The algebraic approach uses the fact that for $u \in \Gamma(g)$ its inverse u^{-1} is proportional to $\beta(u)$. Therefore, up to a normalization $s\beta(s')$ transforms under the tensorial action of $\Gamma(g)$. A decomposition in terms of a basis of the Clifford algebra gives the tensorial pieces of certain rank. In terms of representations we construct a bilinear form on spinors considering induced representations of the opposite Clifford algebra. Given a representation $\gamma : \mathfrak{A} \longrightarrow \mathrm{End}_{\mathbb{F}}(W)$ there is an induced representation γ^T, its "transpose":

$$\begin{aligned} \gamma^T : \mathfrak{A}_{\mathrm{opp}} &\longrightarrow \mathrm{End}_{\mathbb{F}}(W^T) \\ a_{\mathrm{opp}} &\mapsto (\gamma^T)(a_{\mathrm{opp}}) := (\gamma(a))^T : W^T \longrightarrow W^T \\ &\qquad\qquad w^T \mapsto \gamma^T(a_{\mathrm{opp}})(w^T) = w^T \gamma^T(a_{\mathrm{opp}}). \end{aligned} \tag{2.95}$$

This is indeed a representation since

$$\begin{aligned} \gamma^T(a_{\mathrm{opp}} \vee_{\mathrm{opp}} b_{\mathrm{opp}}) &= (\gamma(b_{\mathrm{opp}} \vee a_{\mathrm{opp}}))^T = (\gamma(b)\gamma(a))^T \\ &= \gamma^T(a_{\mathrm{opp}})\gamma^T(b_{\mathrm{opp}}) \qquad \forall\, a_{\mathrm{opp}}, b_{\mathrm{opp}} \in \mathfrak{A}_{\mathrm{opp}}. \end{aligned} \tag{2.96}$$

As we pointed out in (2.50), the main antiautomorphism β can be viewed as connecting the algebra $\mathfrak{A}$ and its opposite $\mathfrak{A}_{\mathrm{opp}}$, so that we may obtain another induced representation $\check{\gamma}$ for $\mathfrak{A}$ by

$$\check{\gamma}(a) := \gamma^T(\beta(a)) = (\gamma(\beta(a)))^T \qquad (a \in \mathfrak{A}), \tag{2.97}$$

where we interpret β once $\mathfrak{A} \xrightarrow{\beta} \mathfrak{A}_{\mathrm{opp}}$ as in (2.50) and then as an antiautomorphism $\mathfrak{A} \xrightarrow{\beta} \mathfrak{A}$ on $\mathfrak{A}$.

Since a bilinear form on spinors can be understood as a linear transformation $B : W \to W^T$, we take B to be a map that intertwines the representations γ and $\check{\gamma}$. Such a map exists if the representation γ is irreducible, whence $\check{\gamma}$ is also irreducible. In this case B is defined up to a constant by

$$B \circ \gamma(a) = \check{\gamma}(a) \circ B \quad \forall\, a \in \mathfrak{A} \qquad \Longleftrightarrow \qquad B\gamma(e_k) = (\gamma(e_k))^T B \quad \forall\, k \in \{1, \dots, n\}. \tag{2.98}$$

We understand B as a bilinear form on W:

$$\begin{aligned} B : W \times W &\to \mathbb{F} \\ (s, s') &\mapsto B(s, s') := (B(s))(s') = \bar{s}s' = s^T B s' \end{aligned} \tag{2.99}$$

both as a map and as its matrix form. $\bar{s} := B(s) = s^T B$ is the adjoint to s with respect to B. Indeed, $B(s, s')$ transforms like a scalar (compare (2.73)):

$$\begin{aligned} B(s, s') \stackrel{\psi_u}{\mapsto} B(u \vee s, u \vee s') &= s^T \gamma(u)^T B \gamma(u) s' = s^T \check{\gamma}(\beta(u)) B \gamma(u) s' \\ &= s^T B \gamma(\beta(u)) \gamma(u) s' = [\beta(u) \vee u]\, s^T B s', \end{aligned} \tag{2.100}$$

if $u = u_1 \vee \cdots \vee u_k \in \Gamma(g)$, $u_1, \dots, u_k \in V$ such that

$$\beta(u) \vee u = g(u_1, u_1) \dots g(u_k, u_k) = 1. \tag{2.101}$$

For $x \in V$, $x \vee s' \stackrel{\psi_u}{\mapsto} u \vee x \vee s' = (u \vee x \vee u^{-1}) \vee (u \vee s')$, hence $B(s, x \vee s')$ also transforms like a scalar. Therefore, a vector y is given by

$$y_k = B(s, e_k \vee s') = s^T B \gamma(e_k) s' \qquad (1 \le k \le n). \tag{2.102}$$

In a similar way, a tensor Y of rank r may be formed:

$$Y_{k_1 \dots k_r} = B(s, e_{k_1} \vee \cdots \vee e_{k_r} \vee s') = s^T B \gamma(e_{k_1}) \dots \gamma(e_{k_r}) s' \qquad (1 \le k_1, \dots, k_r \le n). \tag{2.103}$$

Another bilinear form E may be obtained by replacing the main antiautomorphism β with $\alpha \circ \beta$ which, of course, is an antiautomorphism also. So E is determined up to a constant by

$$E \circ \gamma(a) = \check{\gamma}(\alpha(a)) \circ E \quad \forall\, a \in \mathfrak{A} \quad \Longleftrightarrow \quad E\gamma(e_k) = -(\gamma(e_k))^T E \quad \forall\, k \in \{1, \dots, n\}; \tag{2.104}$$

therefore, for even n

$$E = B\gamma(\eta). \tag{2.105}$$

The condition (2.101) changes to

$$(\alpha \circ \beta)(u) \vee u = (-1)^k g(u_1, u_1) \dots g(u_k, u_k) = 1, \tag{2.106}$$

which reduces to the previous condition for $u \in \Gamma_0(g)$. So both bilinear forms are invariant under the action of normalized elements of $\Gamma_0(g)$.

Both of these bilinear forms may be combined with C to give a sesquilinear form $A : W \to W^\dagger$ on W. We only consider the combination $A := B^* \circ C$ here:

$$\begin{aligned} A \circ \gamma(a) &= B^* \circ C \circ \gamma(a) = B^* \circ \gamma^*(a) \circ C = \gamma^*(\beta(a))^T \circ B^* \circ C \\ &= \gamma^\dagger(\beta(a)) \circ A \qquad \forall\, a \in \mathfrak{A}. \end{aligned} \tag{2.107}$$

$$\Longleftrightarrow \qquad A\gamma(e_k) = \gamma^\dagger(e_k) A \qquad \forall\, k \in \{1, \dots, n\},$$

By a similar argument as in (2.91),

$$\begin{aligned} (A^{-1})^\dagger \circ A \circ \gamma(a) &= (A^{-1})^\dagger \circ \gamma^\dagger(\beta(a)) \circ (A^\dagger \circ (A^{-1})^\dagger) \circ A \\ &= (A^{-1})^\dagger \circ (A \circ \gamma(\beta(a))^\dagger \circ (A^{-1})^\dagger \circ A \\ &= (A^{-1})^\dagger \circ (\gamma^\dagger((\beta \circ \beta)(a)) \circ A)^\dagger \circ (A^{-1})^\dagger \circ A \\ &= ((A^{-1})^\dagger \circ A^\dagger) \circ \gamma(a) \circ A^\dagger \circ A \\ &= \gamma(a) \circ A^\dagger \circ A \qquad \forall\, a \in \mathfrak{A}, \end{aligned} \tag{2.108}$$

we conclude by Schur's Lemma that we can normalize A to satisfy

$$(A^{-1})^{\dagger} \circ A = \mathbf{1}. \tag{2.109}$$

Therefore, A may be assumed to be hermitian. Of course, A may be used to define a spinor adjoint $\overline{s} := A(s) = s^{\dagger}A$ and to construct tensors of various rank as sesquilinear forms of spinors. The condition (2.101) applies also. Which one of these forms is chosen depends on the signature and the physical theory.

In all of our derivations involving C, B, E, and A, we relied on certain properties of matrix multiplication over the field $\mathbb{C}$ (resp. $\mathbb{R}$), namely the fact that transposition is an antiautomorphism and complex conjugation is an automorphism of matrix multiplication. We are about to replace $\mathbb{F}$ by $\mathbb{O}$. Since octonionic multiplication is not commutative and octonionic conjugation is an antiautomorphism, only hermitian conjugation remains as an antiautomorphism of octonionic matrix multiplication. Due to the non-associativity of the octonions even the carrier space W is no longer a vector space, but an "octonionic module". The following section 2.4 will show how to handle these difficulties.

2.4. AN OCTONIONIC REPRESENTATION OF $Cl(8,0)$

In this section we will put the results of sections 2.2 and 2.3 to work and examine the features of octonionic representations of Clifford algebras, considering the example of $Cl(8,0)$. So $V = \mathbb{R}^8$ with a positive definite norm. Let $\{e_0, e_1, \ldots, e_7\}$ be an orthonormal basis of V. Note that we choose indices ranging from 0 to 7 in this section. The octonionic algebra $\mathbb{O}$ is assumed to be given with basis $\{i_0, i_1, \ldots, i_7\}$ obeying the multiplication table (2.23). However, the properties

$$
\begin{aligned}
i_0 &= 1, \\
i_a^2 &= -1 \qquad (1 \le a \le 7), \\
i_a i_b &= -i_b i_a \qquad (1 \le a < b \le 7)
\end{aligned}
\tag{2.110}
$$

rather than the particular multiplication rule, i.e., the particular set P of triples, will be relevant. Furthermore, we identify V and $\mathbb{O}$ as vector spaces by $x^k e_k \mapsto x^k i_k$.

2.4.1. The representation

An octonionic representation $\gamma_{8,0} : Cl(8,0) \to M_2(\mathbb{O})$ is given by

$$
\gamma_{8,0}(e_k) := \begin{pmatrix} 0 & i_k \\ i_k^* & 0 \end{pmatrix} =: \Gamma_k \qquad (0 \le k \le 7) \tag{2.111}
$$

$$
\Longleftrightarrow \qquad \gamma_{8,0}(x) := \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} = x^k \Gamma_k =: \not{x} \qquad (x \in V). \tag{2.112}
$$

The carrier space W of the representation is understood to be $\mathbb{O}^2$, i.e., the set of columns of two octonions, with $\gamma_{8,0}(x)$ acting on it by left multiplication. Therefore, octonionic matrix products are interpreted as being associated to the right and acting on W, i.e., octonionic matrix multiplication is understood to be composition of left multiplication onto W. For example, if we want to verify that (2.112) is a representation, then checking that

$$
\not{x}\not{x} = \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} = \begin{pmatrix} x\,x^* & 0 \\ 0 & x^*\,x \end{pmatrix} = |x|^2 \mathbf{1} = g(x,x)\mathbf{1} \qquad \forall x \in V \tag{2.113}
$$

in accordance with (2.83) is not sufficient. This relationship has to hold even when acting on an element $w = \binom{w_0}{w_1} \in W$:

$$
\begin{aligned}
\not{x}\not{x}w := \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \left(\begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} \right) = \begin{pmatrix} x\,(x^* w_0) \\ x^*(x w_1) \end{pmatrix} = \begin{pmatrix} (x\,x^*) w_0 \\ (x^* x) w_1 \end{pmatrix} \\
= |x|^2 w \qquad \forall w \in W \, \forall x \in V.
\end{aligned}
\tag{2.114}
$$

Thus the alternative property (2.19) of the octonions ensures the validity of the representation.

We need to show that there are no non-trivial invariant subspaces for the representation to be irreducible. We do this in two steps. First, we show that $\binom{1}{0} \in W$ can be mapped to any $w \in W$:

$$(\not{\psi}_1^* + \not{\psi}_0 \not{1})\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0 & w_1^*\\ w_1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0 & w_0\\ w_0^* & 0\end{pmatrix}(\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}) = \begin{pmatrix}w_0\\ w_1\end{pmatrix}. \tag{2.115}$$

Second, we will show that any $0 \neq w \in W$ can be mapped to $\binom{1}{0}$, using the Weyl projections $P_\pm$. If this is so, then there are no non-trivial invariant subspaces of the representation $\gamma_{8,0}$.

Since (2.46) holds for the volume element η, we have for $\Gamma_9 := \gamma_{8,0}(\eta) = \not{i}_0 \not{i}_1 \dots \not{i}_7$

$$\begin{aligned}\Gamma_9 \not{x} &= \begin{pmatrix}0 & i_0(i_1^*(i_2(\dots(i_7^* x)\dots)))\\ i_0^*(i_1(i_2^*(\dots(i_7 x^*)\dots))) & 0\end{pmatrix}\\ = -\not{x}\Gamma_9 &= \begin{pmatrix}0 & -x(i_0^*(i_1(i_2^*(\dots(i_6^* i_7)\dots))))\\ -x^*(i_0(i_1^*(i_2(\dots(i_6 i_7^*)\dots)))) & 0\end{pmatrix} \quad \forall x \in V,\end{aligned} \tag{2.116}$$

hence

$$i_0(i_1(i_2(\dots(i_7 x)\dots))) = x(i_0(i_1(i_2(\dots(i_6 i_7)\dots)))) \qquad \forall x \in \mathbb{O}. \tag{2.117}$$

Since $\Gamma_9^2 = 1$, Γ_9 has eigenvalues ± 1, whence we can find solutions to the equation

$$\Gamma_9 w = \pm w$$

$$\Longleftrightarrow \quad \begin{pmatrix}i_0^*(i_1(i_2^*(\dots(i_7^* w_0)\dots)))\\ i_0(i_1^*(i_2(\dots(i_7 w_1)\dots)))\end{pmatrix} = \begin{pmatrix}-w_0(i_0^*(i_1(i_2^*(\dots(i_6^* i_7)\dots))))\\ -w_1(i_0(i_1^*(i_2(\dots(i_6 i_7^*)\dots))))\end{pmatrix} = \pm\begin{pmatrix}w_0\\ w_1\end{pmatrix}. \tag{2.118}$$

Since a non-trivial solution exists,

$$i_0(i_1(i_2(\ldots(i_7x)\ldots))) = \pm x \qquad \forall x \in \mathbb{O}. \tag{2.119}$$

Which sign is true depends on the specific multiplication rule. With our convention the plus sign applies. In fact, the sign difference corresponds to the two classes of multiplication tables. Since Γ_9 is defined by its action under left multiplication, we have an octonionic Weyl representation:

$$\Gamma_9 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.120}$$

The Weyl projections take the form

$$P_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.121}$$

For any $0 \neq w \in W$, at least one of P_+w or P_-w does not vanish. If $P_+w \neq 0$, then

$$\not{\psi}_0{}^{-1}\not{1}P_+w = \begin{pmatrix} 0 & w_0^{-1} \\ (w_0^{-1})^* & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{2.122}$$

(Note that $\not{1} = \Gamma_0$ corresponds to a vector $e_0 \in V \subseteq Cl(8,0)$ and is to be distinguished from the identity $\gamma(1) = \mathbf{1}$.) If $P_-w \neq 0$, then

$$\not{\psi}_1{}^{-1}P_-w = \begin{pmatrix} 0 & w_0^{-1} \\ (w_0^{-1})^* & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{2.123}$$

This completes the proof that $\gamma_{8,0}$ is irreducible. Since $Cl(8,0)$ is simple, it does not contain any two-sided ideals other than $\{0\}$ and itself, which are also the only candidates for the kernel of any representation of $Cl(8,0)$. Therefore, $\gamma_{8,0}$ is faithful, since it is not trivial. Faithfulness of the representation can also been shown constructively without using the fact that $Cl(8,0)$ is simple. One has to check, for example, if the dimension of the algebra generated by $\{\Gamma_0, \Gamma_1, \ldots, \Gamma_7\}$ is 2^8. Another approach is to construct orthogonal transformations (see [4]), since the Clifford group spans the Clifford algebra. So if the representation obtained for the Clifford group is faithful, then so is the representation for the Clifford algebra.

We chose to rely only on the algebraic properties of the octonions, rather than using the correspondence to a real representation. For completeness, we give the matrices corresponding to left multiplication with respect to our convention:

$$\begin{aligned} \Gamma_0 &= \sigma \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}, & \Gamma_1 &= -\epsilon \otimes \mathbf{1} \otimes \mathbf{1} \otimes \epsilon, \\ \Gamma_2 &= -\epsilon \otimes \tau \otimes \epsilon \otimes \tau, & \Gamma_3 &= -\epsilon \otimes \mathbf{1} \otimes \epsilon \otimes \sigma, \\ \Gamma_4 &= -\epsilon \otimes \epsilon \otimes \mathbf{1} \otimes \tau, & \Gamma_5 &= -\epsilon \otimes \epsilon \otimes \tau \otimes \sigma, \\ \Gamma_6 &= -\epsilon \otimes \sigma \otimes \epsilon \otimes \tau, & \Gamma_7 &= -\epsilon \otimes \epsilon \otimes \sigma \otimes \sigma. \end{aligned} \tag{2.124}$$

Since we have an irreducible representation, we may identify the carrier space W with the space of spinors. So for now we consider elements of $\mathbb{O}^2$ as octonionic spinors. Later in section 2.4.5 we will add a subtle twist to this understanding.

2.4.2. The hermitian conjugate representation and spinor covariants

Since octonionic conjugation is an antiautomorphism of $\mathbb{O}$, the octonionic conjugate of the product of two matrices is not the product of the octonionic conjugates. Matrix transposition requires a commutative multiplication to be an antiautomorphism. Thus only hermitian conjugation, which combines both operations, remains as an antiautomorphism of $M_2(\mathbb{O})$. More precisely, for products of three matrices we need to keep the grouping of the product the same, i.e., under hermitian conjugation left multiplication by a matrix goes to right multiplication by its hermitian conjugate and vice versa. So we can define $\check{\gamma}_{8,0} : Cl(8,0) \to (M_2(\mathbb{O}))^\dagger$ by

$$\check{\gamma}_{8,0}(a) := (\gamma_{8,0}(\beta(a)))^\dagger \qquad (a \in Cl(8,0)). \tag{2.125}$$

This representation acts on the set $W^\dagger = (\mathbb{O}^2)^\dagger$ of rows of two octonions by right multiplication. It is also faithful and irreducible and therefore equivalent to $\gamma_{8,0}$. The isomorphism A intertwining $\gamma_{8,0}$ and $\check{\gamma}_{8,0}$ is given by

$$\begin{aligned} A: \quad W \quad &\to \quad W^\dagger \\ w = \binom{w_0}{w_1} &\mapsto w^\dagger = (w_0^*, w_1^*). \end{aligned} \tag{2.126}$$

Its matrix form is just the identity,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.127}$$

which is verified,

$$A \circ \gamma_{8,0}(a) = \check{\gamma}_{8,0}(a) \circ A \quad \forall\, a \in Cl(8,0) \quad \Longleftrightarrow \quad A\gamma_{8,0}(x) = (\gamma_{8,0}(x))^\dagger \circ A \quad \forall\, x \in V, \tag{2.128}$$

considering $\Gamma_k = (\Gamma_k)^\dagger \quad (0 \le k \le 7)$.

From A we obtain a hermitian form on W:

$$\begin{aligned} &A : W \times W \to \mathbb{R} \\ &(w, z) \mapsto A(w, z) := (A(w))(z) = \mathrm{Re}\, w^\dagger A z = \mathrm{Re}\,(w_0^*, w_1^*) \binom{z_0}{z_1} \\ &\qquad = \mathrm{Re}\,(w_0^* z_0 + w_1^* z_1). \end{aligned} \tag{2.129}$$

The designation "hermitian" is somewhat misleading, since the octonionic representation $\gamma_{8,0}$ is Majorana, i.e., essentially real, which is also the reason for taking the real part above. So the spinor adjoint is given by

$$\overline{w} := A(w) = w^\dagger A = w^\dagger \qquad (w \in W). \tag{2.130}$$

Apart from the scalar, we form tensors as spinor bilinears as in (2.103):

$$Y_{k_1 \ldots k_r} := \mathrm{Re}\, \overline{w} \Gamma_{k_1} \ldots \Gamma_{k_r} z. \tag{2.131}$$

Since the real part of an associator vanishes (2.26) and A is real, we may associate the matrices sandwiched between the two spinors differently:

$$\begin{aligned}\mathrm{Re}\,\overline{w}\Gamma_{k_1}\dots\Gamma_{k_r}z &= \mathrm{Re}\,(w^\dagger A)[\Gamma_{k_1}(\dots(\Gamma_{k_r}z)\dots)]\\ &= \mathrm{Re}\,[(w^\dagger A)\Gamma_{k_1}](\Gamma_{k_2}(\dots(\Gamma_{k_r}z)\dots))\\ &= \mathrm{Re}\,[w^\dagger(A\Gamma_{k_1})](\Gamma_{k_2}(\dots(\Gamma_{k_r}z)\dots))\\ &= \mathrm{Re}\,[(w^\dagger\Gamma^\dagger_{k_1})A](\Gamma_{k_2}(\dots(\Gamma_{k_r}z)\dots))\\ &= \mathrm{Re}\,\overline{\Gamma_{k_1}w}(\Gamma_{k_2}(\dots(\Gamma_{k_r}z)\dots)).\end{aligned} \tag{2.132}$$

Since the real part of a commutator vanishes also, we may cyclicly permute, if a trace is included

$$\begin{aligned}\mathrm{Re}\,\overline{w}\Gamma_{k_1}\dots\Gamma_{k_r}z &= \mathrm{Re}\,\mathrm{tr}\,(\overline{w}(\Gamma_{k_1}(\dots(\Gamma_{k_r}z)\dots))) = \mathrm{Re}\,\mathrm{tr}\,((\Gamma_{k_1}(\dots(\Gamma_{k_r}z)\dots))\overline{w})\\ &= \mathrm{Re}\,\mathrm{tr}\,((\Gamma_{k_2}(\dots(\Gamma_{k_r}z)\dots)\overline{w})\Gamma_{k_1}).\end{aligned} \tag{2.133}$$

For the vector covariant, we have a particular expression

$$\begin{aligned}y_k := \mathrm{Re}\,\overline{w}\Gamma_k z &= \mathrm{Re}\,(w_0^*, w_1^*)\begin{pmatrix}0 & i_k\\ i_k^* & 0\end{pmatrix}\begin{pmatrix}z_0\\ z_1\end{pmatrix} = \mathrm{Re}\,(w_0^* i_k z_1 + w_1^* i_k^* z_0)\\ &= \mathrm{Re}\,(i_k z_1 w_0^* + z_0 w_1^* i_k^*) = \mathrm{Re}\,(w_0 z_1^* i_k^* + z_0 w_1^* i_k^*)\\ &= (w_0 z_1^* + z_0 w_1^*)_k\,,\end{aligned} \tag{2.134}$$

where we used once for part of the expression that the real part does not change under octonionic conjugation. So we can express the k-th component of y by the k-th component of an octonionic product, which allows us to write $\not{y}$ without the use of the matrix representations of the basis elements:

$$\begin{aligned}\not{y} &= \begin{pmatrix}0 & y\\ y^* & 0\end{pmatrix} = \Gamma^k \mathrm{Re}\,\overline{w}\Gamma_k z\\ &= \begin{pmatrix}0 & w_0 z_1^* + z_0 w_1^*\\ (w_0 z_1^* + z_0 w_1^*)^* & 0\end{pmatrix}.\end{aligned} \tag{2.135}$$

2.4.3. Orthogonal transformations

From section 2.3.2 we know the action of the Clifford group on vectors (2.62) and spinors (2.73). The condition (2.101) shows how to divide out $\mathbb{R}^*$ to obtain the orthogonal group. So elements of V satisfying

$$\beta(u) \vee u = 1 \qquad \Longleftrightarrow \qquad u \vee u = g(u,u) = |u|^2 = 1 \tag{2.136}$$

generate the orthogonal transformations via

$$\not{x}' = (\gamma \circ \phi_u)(x) = \not{u}\not{x}\not{u} = \begin{pmatrix} 0 & ux^*u \\ u^*xu^* & 0 \end{pmatrix}, \tag{2.137}$$

$$w' = \psi_u(w) = \not{u}w = \begin{pmatrix} uw_1 \\ u^*w_0 \end{pmatrix}. \tag{2.138}$$

The Moufang (2.10) identities ensure that (2.137) is unambiguous and even holds under the action of left multiplication, which can be seen in the example, $(x \vee w)' = x' \vee w'$:

$$\begin{aligned} \not{x}'w' &= \begin{pmatrix} 0 & ux^*u \\ u^*xu^* & 0 \end{pmatrix} \begin{pmatrix} uw_1 \\ u^*w_0 \end{pmatrix} = \begin{pmatrix} (u^*xu^*)(uw_1) \\ (ux^*u)(u^*w_0) \end{pmatrix} \\ &= \begin{pmatrix} u^*(x(u^*(uw_1))) \\ u(x^*(u(u^*w_0))) \end{pmatrix} = \begin{pmatrix} u^*(x((u^*u)w_1)) \\ u(x^*((uu^*)w_0)) \end{pmatrix} \\ &= |u|^2 \begin{pmatrix} u^*(xw_1) \\ u(x^*w_0) \end{pmatrix} = \not{u}(\not{x}w) \\ &= (\not{x}w)'. \end{aligned} \tag{2.139}$$

The third Moufang identity guarantees that the vector covariant (2.135) of two spinors transform correctly:

$$
\begin{aligned}
\not{y}' &= \begin{pmatrix} 0 & uy^*u \\ u^*yu^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & u(w_0z_1^* + z_0w_1^*)^*u \\ u^*(w_0z_1^* + z_0w_1^*)u^* & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & (u^*(w_0z_1^* + z_0w_1^*)u^*)^* \\ u^*(w_0z_1^* + z_0w_1^*)u^* & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & [(u^*w_0)(z_1^*u^*) + (u^*z_0)(w_1^*u^*)]^* \\ (u^*w_0)(z_1^*u^*) + (u^*z_0)(w_1^*u^*) & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & [(u^*w_0)(uz_1)^* + (u^*z_0)(uw_1)^*]^* \\ (u^*w_0)(uz_1)^* + (u^*z_0)(uw_1)^* & 0 \end{pmatrix} \\
&= \Gamma^k \mathrm{Re}\, \overline{w'} \Gamma_k z'.
\end{aligned}
\tag{2.140}
$$

According to (2.72), simple orthogonal transformations are generated by pairs $(u, v) \in V \times V$, where we take $v = e_0$ fixed and $|u|^2 = 1$:

$$
\not{x}' = (\gamma_{8,0} \circ \phi_{(u \vee v)})(x) = \not{u}\not{1}\not{x}\not{1}\not{u} = \begin{pmatrix} 0 & uxu \\ u^*x^*u^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & uxu \\ (uxu)^* & 0 \end{pmatrix}, \tag{2.141}
$$

$$
w' = \psi_{(u \vee v)}(w) = \not{u}\not{1}w = \begin{pmatrix} uw_0 \\ u^*w_1 \end{pmatrix}. \tag{2.142}
$$

Choosing the fixed vector to be e_0 allows significant simplification, since its representation Γ_0 is real. How to construct any orthogonal transformation from these generators is thoroughly explained in [4]. These transformation properties imply that the definition of the spinor covariants in section 2.4.2 is consistent. For example,

$$
\mathrm{Re}\, \overline{w'}\not{x}'z' = \mathrm{Re}\, \overline{\not{u}\not{1}w}\, \not{u}\not{1}\not{x}\not{1}\not{u}\, \not{u}\not{1}z = \mathrm{Re}\, w^\dagger \not{u}^\dagger \not{1}^\dagger A \not{u}\not{1}\, \not{x}\, z = \mathrm{Re}\, w^\dagger A \not{1}\not{u}\not{u}\not{1}\, \not{x}\, z = \mathrm{Re}\, \overline{w}\, \not{x}\, z\,. \tag{2.143}
$$

2.4.4. Related representations using the opposite octonionic algebra $\mathbb{O}_{\mathrm{opp}}$

As pointed out in section 2.4.2, transposition and octonionic conjugation are not (anti-)automorphisms of octonionic matrix multiplication. However, we can find (anti-)isomorphisms to matrix algebras over the opposite octonionic algebra $\mathbb{O}_{\text{opp}}$. We define the octonionic conjugate representation γ^* of an octonionic representation $\gamma : \mathfrak{A} \to M_l(\mathbb{O})$ by

$$\begin{aligned} \gamma^* : \mathfrak{A} &\to M_l(\mathbb{O}^*_{\text{opp}}) \\ a &\mapsto \gamma^*(a) := (\gamma(a))^*_{\text{opp}}. \end{aligned} \tag{2.144}$$

Octonionic products are now to be evaluated in the opposite algebra as it is indicated in the following examples. First we consider the action of $\gamma^*_{8,0}(x)$ for $x \in V$ on an element w^* of the carrier space W^*_{opp}

$$\begin{aligned} \gamma^*_{8,0}(x) w^*_{\text{opp}} &= \left(\begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix}^* \begin{pmatrix} w_0 \\ w_1 \end{pmatrix}^* \right)_{\text{opp}} = \left(\begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix} \begin{pmatrix} w_0^* \\ w_1^* \end{pmatrix} \right)_{\text{opp}} \\ &= \begin{pmatrix} w_1^* x^* \\ w_0^* x \end{pmatrix} \\ = (\gamma_{8,0}(x) w)^* &= \begin{pmatrix} x w_1 \\ x^* w_0 \end{pmatrix}^* = \begin{pmatrix} (x w_1)^* \\ (x^* w_0)^* \end{pmatrix} \end{aligned} \tag{2.145}$$

So in this representation the action on the carrier space is effectively right multiplication by octonions.

We check that $\gamma^*_{8,0}$ is indeed a representation. Let $u, v \in V$, then

$$\begin{aligned} \gamma^*_{8,0}(u)\gamma^*_{8,0}(v) &= \begin{pmatrix} 0 & u \\ u^* & 0 \end{pmatrix}^*_{\text{opp}} \begin{pmatrix} 0 & v \\ v^* & 0 \end{pmatrix}^*_{\text{opp}} = \left(\begin{pmatrix} 0 & u^* \\ u & 0 \end{pmatrix} \begin{pmatrix} 0 & v^* \\ v & 0 \end{pmatrix} \right)_{\text{opp}} \\ &= \begin{pmatrix} v u^* & 0 \\ 0 & v^* u \end{pmatrix} \\ = \gamma^*_{8,0}(u \vee v) &= (\gamma_{8,0}(u \vee v))^* = (\not{u}\not{v})^* \\ &= \begin{pmatrix} u v^* & 0 \\ 0 & u^* v \end{pmatrix}^*. \end{aligned} \tag{2.146}$$

In both cases the subscript "opp" indicates that the remaining products are to be done in the opposite octonionic algebra. However the final result is to be interpreted as an element of W^*_{opp} (resp. $M_2(\mathbb{O}^*_{\text{opp}})$).

The map that intertwines $\gamma^*_{8,0}$ and $\gamma_{8,0}$ is

$$\begin{aligned} C : W &\to W^*_{\mathrm{opp}} \\ w &\mapsto C(w) := \Gamma_0 w_{\mathrm{opp}}, \end{aligned} \tag{2.147}$$

since

$$\Gamma_0 \Gamma_k = (\Gamma_k)^* \Gamma_0 \qquad (0 \le k \le 7). \tag{2.148}$$

This map gives rise to an operation on W analogous to charge conjugation:

$$w_C := C(w)^* = \Gamma_0 w^* = \begin{pmatrix} w_1^* \\ w_0^* \end{pmatrix} \in W. \tag{2.149}$$

w_C transforms correctly:

$$\begin{aligned} (w_C)' = \not{u} w_C &= \begin{pmatrix} 0 & u \\ u^* & 0 \end{pmatrix} \begin{pmatrix} w_1^* \\ w_0^* \end{pmatrix} \\ &= \begin{pmatrix} u w_0^* \\ u^* w_1^* \end{pmatrix} = \begin{pmatrix} w_0 u^* \\ w_1 u \end{pmatrix}^* \\ &= \begin{pmatrix} u^* w_0 \\ u w_1 \end{pmatrix}^*_{\mathrm{opp}} = \left(\left[\begin{pmatrix} 0 & u^* \\ u & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} \right]_{\mathrm{opp}} \right)^* \\ &= \left(\left[\gamma^*_{8,0}(u) \Gamma_0 w \right]_{\mathrm{opp}} \right)^* \\ &= \Gamma_0 \left([\gamma_{8,0}(u) w]_{\mathrm{opp}} \right)^*. \end{aligned} \tag{2.150}$$

However, the opposite octonionic algebra may not be bypassed:

$$(w_C)' \neq (C(w'))^* = \left[\Gamma_0 \begin{pmatrix} u w_1 \\ u^* w_0 \end{pmatrix} \right]^* . = \begin{pmatrix} u^* w_0 \\ u w_1 \end{pmatrix}^* . \tag{2.151}$$

Related to matrix transposition we obtain another representation $\check{\gamma}$ involving $\mathbb{O}_{\mathrm{opp}}$:

$$\begin{aligned} \check{\gamma} : \mathfrak{A} &\to M_l^T(\mathbb{O}_{\mathrm{opp}}) \\ a &\mapsto \check{\gamma}(a) := (\gamma(\beta(a)))^T_{\mathrm{opp}} : W^T_{\mathrm{opp}} \to W^T_{\mathrm{opp}} \\ & \qquad\qquad w^T \mapsto \check{\gamma}(a)(w^T) = \left(w^T (\gamma(\beta(a)))^T \right)_{\mathrm{opp}} . \end{aligned} \tag{2.152}$$

The verification of $\check{\gamma}(a \vee b) = \check{\gamma}(a)\check{\gamma}(b)$ is another exercise in applying opposite algebras:

$$\begin{aligned}\check{\gamma}(a \vee b) &= (\gamma(\beta(a \vee b)))^T = (\gamma(\beta(b) \vee \beta(a)))^T \\ &= (\gamma(\beta(b))\gamma(\beta(a)))^T = \left((\gamma(\beta(a)))^T (\gamma(\beta(b)))^T\right)_{\mathbf{opp}} \\ &= \check{\gamma}(a)\check{\gamma}(b).\end{aligned} \tag{2.153}$$

The map that intertwines $\check{\gamma}_{8,0}$ and $\gamma_{8,0}$ is

$$\begin{aligned} B : W &\rightarrow W^T_{\mathbf{opp}} \\ w &\mapsto B(w) := w^T_{\mathbf{opp}}\Gamma_0, \end{aligned} \tag{2.154}$$

since

$$\Gamma_0\Gamma_k = (\Gamma_k)^T\Gamma_0 \qquad (0 \leq k \leq 7). \tag{2.155}$$

2.4.5. Octonionic spinors as elements of minimal left ideals

In this section we take a different perspective on octonionic spinors regarding them as elements of a minimal left ideal which is generated by a certain primitive idempotent. The choice of an idempotent will turn out to be equivalent to the choice of a basis of the carrier space of the representation, which may be understood as a change of the multiplication rule of the octonions.

In a real or complex representation $\gamma : \mathfrak{A} \rightarrow \mathrm{End}_{\mathbb{F}}(W, W)$ of dimension l an idempotent is given by an $l \times l$-matrix Q satisfying the minimal polynomial $Q(Q-1) = 0$. Therefore, Q can be diagonalized with eigenvalues 0 and 1. If the representation is onto and the idempotent is primitive, then Q is of rank 1 and there is a transformation such that Q takes the form

$$Q = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix} . \tag{2.156}$$

So for a surjective representation a primitive idempotent is represented by a matrix of the form

$$Q = q\,p^T, \quad p^T q = 1 \qquad (q, p \in W). \tag{2.157}$$

The action of the Clifford algebra on the minimal left ideal $J = \mathfrak{A}_\vee Q$ generated by Q, is determined by q alone. So the relevant choices of primitive idempotents are given by the choices for q. The choice of a basis for J is still arbitrary at this point, for the octonionic case, however, there is a connection between the choice of q and a multiplication rule.

In terms of the octonionic representation $\gamma_{8,0}$ we have $q = \binom{q_0}{q_1} \in \mathbb{O}^2$. For q to correspond to an even primitive idempotent Q, one of its components has to vanish. We may also normalize q. So let $q = \binom{\rho}{0}$ with $|\rho|^2 = 1$. (A vanishing upper component leads to similar results.) Then the following choice of octonionic spinor components s_0 and s_1 for a spinor s

$$s := (\not{s}_1 + \not{s}_0 \Gamma_0)\, q = \begin{pmatrix} s_0 & s_1 \\ s_1^* & s_0^* \end{pmatrix} \begin{pmatrix} \rho \\ 0 \end{pmatrix} = \begin{pmatrix} s_0 \rho \\ s_1^* \rho \end{pmatrix} \tag{2.158}$$

is up to octonionic conjugation the only one that involves only one left multiplication by an octonion. Here we will actually consider both s_1 and its conjugate s_1^* as new spinor component. But in section 2.6, s_1 will turn out to be the more convenient choice. Obviously s_0 and s_1 parametrize J. How does the Clifford algebra act in terms of the new spinor components? For $x \in V$

$$
\begin{aligned}
s' = \not{x} s = \begin{pmatrix} x(s_1^*\rho) \\ x^*(s_0\rho) \end{pmatrix} &= \begin{pmatrix} s_0'\rho \\ s_1^{*\prime}\rho \end{pmatrix} \\
\Longrightarrow \quad s_0' = x(s_1^*\rho)\rho^*, \quad & s_1' = \rho((\rho^* s_0^*)x),
\end{aligned} \tag{2.159}
$$

which leads to two other versions of the "X-product" (2.34) with $X = \rho$:

$$
\begin{aligned}
s_0' &= [x(s_1^*\rho)]\rho^* = [(x\rho\rho^*)(s_1^*\rho)]\rho^* = [((x\rho)\rho^*)(s_1^*\rho)]\rho^* = (x\rho)[\rho^*(s_1^*\rho)\rho^*], \\
&= (x\rho)(\rho^* s_1^*) = x \underset{\rho}{\circ} s_1^*, \\
s_1' &= \rho[(\rho^* s_0^*)x] = \rho[(\rho^* s_0^*)(\rho\rho^* x)] = \rho[(\rho^* s_0^*)(\rho(\rho^* x))] = [\rho(\rho^* s_0^*)\rho](\rho^* x), \\
&= (s_0^*\rho)(\rho^* x) = s_0^* \underset{\rho}{\circ} x, \\
s_1'^{\,*} &= s_1^{*\prime} = (s_0^* \underset{\rho}{\circ} x)^* = x^* \underset{\rho}{\circ} s_0, \\
\begin{pmatrix} s_0' \\ s_1'^{\,*} \end{pmatrix} &= \not{x} \underset{\rho}{\circ} \begin{pmatrix} s_0 \\ s_1^* \end{pmatrix}.
\end{aligned} \tag{2.160}
$$

Therefore, switching to the new spinor components s_0 and s_1^* is equivalent to replacing the original octonionic product with the "ρ-product". We confirm this result for the scalar formed out of two spinors (compare (2.129)):

$$
\begin{aligned}
\operatorname{Re} \bar{s} s' &= \operatorname{Re} \begin{pmatrix} \rho^* s_0^* & \rho^* s_1 \end{pmatrix} \begin{pmatrix} s_0'\rho \\ s_1'^{\,*}\rho \end{pmatrix} = \operatorname{Re}[(\rho^* s_0^*)(s_0'\rho) + (\rho^* s_1)(s_1'^{\,*}\rho)] \\
&= \operatorname{Re}[(s_0'\rho)(\rho^* s_0^*) + (s_1'^{\,*}\rho)(\rho^* s_1)] = \operatorname{Re}(s_0' \underset{\rho}{\circ} s_0^* + s_1'^{\,*} \underset{\rho}{\circ} s_1) \\
&= \operatorname{Re}(s_0' s_0^* + s_1'^{\,*} s_1) = \operatorname{Re}(s_0^* \underset{\rho}{\circ} s_0' + s_1 \underset{\rho}{\circ} s_1'^{\,*})
\end{aligned} \tag{2.161}
$$

as well as the vector (compare (2.135))

$$
\begin{aligned}
\Gamma^k \operatorname{Re} \bar{s} \Gamma_k s' &= \begin{pmatrix} 0 & (s_0\rho)(\rho^* s_1') + (s_0'\rho)(\rho^* s_1) \\ [(s_0\rho)(\rho^* s_1') + (s_0'\rho)(\rho^* s_1)]^* & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & s_0 \underset{\rho}{\circ} s_1' + s_0' \underset{\rho}{\circ} s_1 \\ [s_0 \underset{\rho}{\circ} s_1' + s_0' \underset{\rho}{\circ} s_1]^* & 0 \end{pmatrix}.
\end{aligned} \tag{2.162}
$$

Of course, orthogonal transformations, as described in section 2.4.3, induce a change of basis on the spinor space also. The corresponding change of the octonionic multiplication rule is more complex since the real part is no longer fixed (compare section 2.2.3).

2.5. OTHER OCTONIONIC REPRESENTATIONS

In this section, we point out the constructions of octonionic representations related to $\gamma_{8,0}$. We follow the program outlined in section 2.3.3. First we shrink the representation of $Cl(8,0)$ to obtain one of $Cl_0(8,0) \cong Cl(0,7)$ and further of $Cl(0,6)$. Then we look at the extension to a representation of $Cl(9,1)$, which is of particular importance, since it applies to superstring and superparticle models.

2.5.1. $Cl_0(8,0)$ and $Cl(0,7)$

Restricting the representation $\gamma_{8,0}$ to $Cl_0(8,0) \cong Cl_0(0,8)$ produces a faithful representation with the generators

$$\Gamma_0\Gamma_k = \gamma_{8,0}(e_0 \vee e_k) = \begin{pmatrix} i_k & 0 \\ 0 & i_k^* \end{pmatrix} = \begin{pmatrix} i_k & 0 \\ 0 & -i_k \end{pmatrix} \qquad (1 \leq k \leq 7). \tag{2.163}$$

So $Cl_0(8,0)$ is represented by diagonal matrices, i.e., this representation decomposes into two irreducible representations given by the two elements on the diagonal. By the isomorphism $Cl_0(8,0) \cong Cl(0,7)$ (2.93), these two are also irreducible representations $\gamma_{0,7}^{\pm} : Cl(0,7) \to M_1(\mathbb{O}) = \mathbb{O}$,

$$\gamma_{0,7}^{\pm}(e_k) := \pm i_k \qquad (1 \leq k \leq 7) \tag{2.164}$$

$$\Longleftrightarrow \qquad \gamma_{0,7}^{\pm}(x) := \pm x = \pm \mathrm{Im}\, x \qquad (x \in V = \mathbb{R}^7). \tag{2.165}$$

So we identify $V = \mathbb{R}^7$ with the purely imaginary subspace of the octonions $\mathrm{Im}\,\mathbb{O}$. A faithful representation of $Cl(0,7)$ is found by letting $\gamma_{0,7}(e_k) = \Gamma_0\Gamma_k$ in (2.163):

$$\gamma_{0,7} := \gamma_{0,7}^{+} \oplus \gamma_{0,7}^{-} \qquad \Longleftrightarrow \qquad \gamma_{0,7}(a) = \begin{pmatrix} \gamma_{0,7}^{+}(a) & 0 \\ 0 & \gamma_{0,7}^{-}(a) \end{pmatrix}. \tag{2.166}$$

A hermitian form $A' : \mathbb{O}^{\pm} \to \mathbb{O}^{\pm\,\dagger}$ on the carrier space of an irreducible representation is given by

$$A'(w) := w^{*} \tag{2.167}$$

with the property

$$A'\gamma_{0,7}^{\pm}(e_k) = -\gamma_{0,7}^{\pm\,\dagger}(e_k)A' = -\left(\gamma_{0,7}^{\pm}(e_k)\right)^{*} \qquad (1 \leq k \leq 7). \tag{2.168}$$

Thus the form A' intertwines $\gamma_{0,7}^{\pm}$ and $\gamma_{0,7}^{\pm\,\dagger} \circ \alpha \circ \beta$:

$$A' \circ \gamma_{0,7}^{\pm}(a) = \left(\gamma_{0,7}^{\pm}((\alpha \circ \beta)(a))\right)^{\dagger} \circ A' \qquad (a \in Cl(0,7)). \tag{2.169}$$

There is no sesquilinear form satisfying (2.107) on a carrier space of the irreducible representation. However, one can intertwine $\gamma_{0,7}^{+}$ and $\gamma_{0,7}^{-}$ to obtain such a form on the carrier space $2\mathbb{O} = \mathbb{O}^{+} \oplus \mathbb{O}^{-}$ of the faithful representation that swaps the two copies $\mathbb{O}^{+}$ and $\mathbb{O}^{-}$ of $\mathbb{O}$ since

$$\gamma_{0,7}^{\pm}(a) = \left(\gamma_{0,7}^{\mp}(\beta(a))\right)^{*} \qquad (a \in Cl(0,7)). \tag{2.170}$$

A, defined by

$$A(w^{+} \oplus w^{-}) := w^{-\,*} \oplus w^{+\,*} = \overline{w} \qquad \Longleftrightarrow \qquad A := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma, \tag{2.171}$$

satisfies

$$\begin{aligned} & A \circ \gamma_{0,7}(a) = \gamma_{0,7}^{\dagger}(\beta(a)) \circ A \qquad (a \in Cl(0,7)) \\ \Longleftrightarrow \qquad & A\gamma_{0,7}(e_k) = \gamma_{0,7}^{\dagger}(e_k)A \qquad (1 \leq k \leq 7). \end{aligned} \tag{2.172}$$

Simple orthogonal transformations are generated by unit vectors $u \in \operatorname{Im}\mathbb{O}$, $|u|^2 = -u^2 = 1$ via

$$x' = (\gamma^{\pm}_{0,7} \circ \phi_u)(x) = (\pm u)x(\pm u)^{-1} = uxu^* = -uxu, \tag{2.173}$$

$$w^{\pm\prime} = \psi_u(w) = \pm uw. \tag{2.174}$$

Since the real part of u vanishes, $u^{-1} = -u$. Therefore, the transformations have the same form as (2.137) and (2.138) up to signs and the Moufang identities ensure the compatibility of the spinor and vector transformations as before. As is seen from (2.66), improper rotations, for example, inversion of $\mathbb{R}^7$, $x \mapsto -x = x^*$, is not described by the action of the Clifford group for odd n. In fact, inversion is equivalent to octonionic conjugation or switching from $\gamma^{\pm}_{0,7}$ to $\gamma^{\mp}_{0,7}$. In order to implement inversion we need to use the faithful representation:

$$\begin{gathered} \not{x}' = -\epsilon\not{x}\epsilon = -\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\not{x}, \\ w' = \begin{pmatrix} w^{+\prime} \\ w^{-\prime} \end{pmatrix} = \epsilon w' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} w^{+} \\ w^{-} \end{pmatrix} = \begin{pmatrix} w^{-} \\ -w^{+} \end{pmatrix}. \end{gathered} \tag{2.175}$$

The transformation preserves scalars:

$$\overline{w}\not{x}z = w^{\dagger}\sigma(-\epsilon)(-\epsilon)\not{x}\epsilon\epsilon z = (w^{\dagger}\epsilon\sigma)[-\epsilon\not{x}\epsilon](\epsilon z) = \overline{w}'\not{x}'z'. \tag{2.176}$$

2.5.2. $Cl_0(0,7)$ and $Cl(0,6)$

Shrinking a representation of $Cl(0,7)$ further leads to the smallest Clifford algebra that has the octonions as a natural carrier space for a representation. Both irreducible representations $\gamma^{+}_{0,7}$ and $\gamma^{-}_{0,7}$ agree on the even Clifford algebra $Cl_0(0,7) \cong Cl_0(7,0)$. Their restriction is an irreducible representation given by the generators

$$\gamma^{\pm}_{0,7}(e_k \vee e_7) = i_k i_7 \qquad (1 \leq k \leq 6), \tag{2.177}$$

which act by successive left multiplication on the carrier space $W = \mathbb{O}$. Again by the isomorphism $Cl_0(0,7) \cong Cl(0,6)$ (2.93), we obtain a faithful and irreducible representation of $Cl(0,6)$, $\gamma_{0,6} : Cl(0,6) \to M_1(\mathbb{O}) = \mathbb{O}$,

$$\gamma_{0,6}(e_k) := i_k i_7 \qquad (1 \leq k \leq 6), \tag{2.178}$$

$$\Longleftrightarrow \qquad \gamma_{0,6}^{\pm}(x) := x i_7 \qquad (x \in V = \mathbb{R}^6). \tag{2.179}$$

$V = \mathbb{R}^6$ is identified with the imaginary subspace of $\mathbb{O}$ with vanishing 7-component, $\{x \in \operatorname{Im}\mathbb{O} : x^7 = 0\}$. The volume form η is represented by

$$\gamma_{0,6}(\eta) = \gamma_{0,6}(e_1 \vee e_2 \vee \cdots \vee e_6) = i_1 i_7 i_2 i_7 \ldots i_6 i_7 = -i_1 i_2 \ldots i_6 = i_7, \tag{2.180}$$

according to (2.119). A hermitian form $A' : \mathbb{O} \to \mathbb{O}^{\dagger}$ is given by

$$A'(w) := w^*. \tag{2.181}$$

Orthogonal transformations are generated by unit vectors $u \in \mathbb{R}^6$, $|u|^2 = -u^2 = 1$ via

$$x' = (\gamma_{0,6} \circ \phi_u)(x) = (u(i_7 x i_7)u) \tag{2.182}$$

$$w' = \psi_u(w) = u(i_7 w) \tag{2.183}$$

Since these transformations have the same structure as the simple orthogonal transformations for $V = \mathbb{R}^8$, the Moufang identities ensure their compatibility and their validity under the interpretation of left multiplication. Since $\gamma_{0,6}$ is faithful and irreducible and $Cl(0,6)$ is a 2^6-dimensional algebra, we conclude from this section that left multiplication by octonions generates a 64-dimensional algebra isomorphic to $M_8(\mathbb{R})$.

2.5.3. $Cl(9,1)$

In this section we will give a little more detail because of the frequent use of $Cl(9,1)$ in supersymmetric models. Starting from $Cl(8,0)$, we do a Cartan extension (2.84) to obtain a representation of $Cl(9,1)$, $\gamma_{9,1} : Cl(9,1) \to M_4(\mathbb{O})$, given by the generators

$$
\begin{aligned}
\gamma_{9,1}(e_k) &:= \sigma \otimes \gamma_{8,0}(e_k) = \begin{pmatrix} 0 & \Gamma_k \\ \Gamma_k & 0 \end{pmatrix} \quad (0 \le k \le 7), \\
\gamma_{9,1}(e_8) &:= \sigma \otimes \gamma_{8,0}(\eta) = \begin{pmatrix} 0 & \tau \\ \tau & 0 \end{pmatrix}, \\
\gamma_{9,1}(e_{-1}) &:= -\epsilon \otimes \gamma_{8,0}(1) = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix},
\end{aligned}
\tag{2.184}
$$

or equivalently by

$$
\begin{aligned}
\gamma_{9,1}(x) &:= \not{x} = x^\mu \gamma_\mu = \begin{pmatrix} \mathbf{0} & \mathbf{X} \\ \tilde{\mathbf{X}} & \mathbf{0} \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{0} & \begin{pmatrix} x^+ & x \\ x^* & x^- \end{pmatrix} \\ \begin{pmatrix} -x^- & x \\ x^* & -x^+ \end{pmatrix} & \mathbf{0} \end{pmatrix},
\end{aligned}
\tag{2.185}
$$

where we defined

$$
\begin{aligned}
\mathbf{X} &:= x^\mu \Gamma_\mu = \begin{pmatrix} x^+ & x \\ x^* & x^- \end{pmatrix}, \qquad \Gamma_8 := \tau,\ \Gamma_{-1} := \mathbf{1},\ x_\pm := x_{-1} \pm x_8, \\
\tilde{\mathbf{X}} &:= x^\mu \tilde{\Gamma}_\mu = \begin{pmatrix} -x^- & x \\ x^* & -x^+ \end{pmatrix}, \quad \tilde{\Gamma}_\mu := \begin{cases} \Gamma_\mu, & (0 \le \mu \le 8) \\ -\Gamma_{-1}, & (\mu = -1) \end{cases}, \\
\gamma_\mu &:= \gamma_{9,1}(e_\mu) = \begin{pmatrix} \mathbf{0} & \Gamma_\mu \\ \tilde{\Gamma}_\mu & \mathbf{0} \end{pmatrix} \quad (-1 \le \mu \le 8).
\end{aligned}
\tag{2.186}
$$

(Labeling the basis elements of $V = \mathbb{R}^{10}$ by indices ranging from -1 to 8, allows us to keep the notation we developed for $\gamma_{8,0}$.) The representation $\gamma_{9,1}$ is Weyl, since the volume element $\eta = e_{-1} \vee e_0 \vee \cdots \vee e_8$ is represented by

$$
\gamma_{9,1}(\eta) = \tau \otimes \mathbf{1} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} = \gamma_{-1}\gamma_0 \ldots \gamma_8 =: \gamma_{11}.
\tag{2.187}
$$

The Weyl projections (2.86) take the form

$$
P_+ = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \qquad P_- = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}.
\tag{2.188}
$$

We denote an element $w \in W = \mathbb{O}^4$ of the carrier space by its Weyl projections

$$w_\pm := P_\pm w \in \mathbb{O}^2, \tag{2.189}$$

where we discard the two vanishing components of $w_\pm$. The identity

$$\not{x}\not{x} = x^\mu x_\mu \mathbf{1} \quad \Longleftrightarrow \quad \begin{pmatrix} \mathbf{X}\tilde{\mathbf{X}} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{X}}\mathbf{X} \end{pmatrix} = x^\mu x_\mu \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, \tag{2.190}$$

holds under left multiplication because of the alternative property (2.19), since only one full octonion x and its conjugate are contained in $\mathbf{X}$ and $\tilde{\mathbf{X}}$. Noting that

$$\tilde{\mathbf{X}} = \mathbf{X} - (\mathrm{tr}\,(\mathbf{X}))\,\mathbf{1}, \tag{2.191}$$

it follows that

$$\mathbf{X}\tilde{\mathbf{X}} = \mathbf{X}^2 - (\mathrm{tr}\,(\mathbf{X}))\,\mathbf{X} = \tilde{\mathbf{X}}\mathbf{X} = -\det\mathbf{X}\,\mathbf{1} = x^\mu x_\mu\,\mathbf{1}, \tag{2.192}$$

since the characteristic polynomial for a hermitian 2×2-matrix A is $p_A(\lambda) = \lambda^2 - \mathrm{tr}\,(A)\lambda + \det A$. Polarizing (2.192), we get

$$\begin{aligned} & 2x_\mu y^\mu\,\mathbf{1} = \mathbf{X}\tilde{\mathbf{Y}} + \mathbf{Y}\tilde{\mathbf{X}} = \tilde{\mathbf{X}}\mathbf{Y} + \tilde{\mathbf{Y}}\mathbf{X} \\ \Longleftrightarrow \quad & 2g_{\mu\nu}\,\mathbf{1} = \Gamma_\mu\tilde{\Gamma}_\nu + \Gamma_\nu\tilde{\Gamma}_\mu = \tilde{\Gamma}_\mu\Gamma_\nu + \tilde{\Gamma}_\nu\Gamma_\mu\,. \end{aligned} \tag{2.193}$$

To extract components, we have the familiar formulas involving traces:

$$x_\mu = \frac{1}{4}\mathrm{Re}\,\mathrm{tr}\,(\not{x}\gamma_\mu) = \frac{1}{4}\mathrm{Re}\,\mathrm{tr}\left(\mathbf{X}\tilde{\Gamma}_\mu + \tilde{\mathbf{X}}\Gamma_\mu\right) = \frac{1}{2}\mathrm{Re}\,\mathrm{tr}\left(\mathbf{X}\tilde{\Gamma}_\mu\right) = \frac{1}{2}\mathrm{Re}\,\mathrm{tr}\left(\tilde{\mathbf{X}}\Gamma_\mu\right). \tag{2.194}$$

Considering

$$\gamma_\mu = \begin{cases} \gamma_\mu, & (\mu \neq -1) \\ -\gamma_\mu, & (\mu = -1) \end{cases}, \tag{2.195}$$

a hermitian form A is given by

$$A(w) := w^\dagger A = w^\dagger \gamma_{11}\gamma_{-1} = \begin{pmatrix} w_+^\dagger & w_-^\dagger \end{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} w_-^\dagger & w_+^\dagger \end{pmatrix} =: \overline{w}\,. \tag{2.196}$$

So the scalar covariant formed out of $w, z \in W$ is

$$A(w,z) = \mathrm{Re}\,\overline{w}z = \mathrm{Re}\,\begin{pmatrix} w_-^\dagger & w_+^\dagger \end{pmatrix} \begin{pmatrix} z_+ \\ z_- \end{pmatrix} = \mathrm{Re}\,(w_-^\dagger z_+ + w_+^\dagger z_-), \qquad (2.197)$$

which only involves terms combining spinors of opposite chirality. For the vector covariant y, we obtain

$$\begin{aligned} y_\mu &:= \mathrm{Re}\,\overline{w}\gamma_\mu z \\ &= \mathrm{Re}\left[\begin{pmatrix} w_-^\dagger & w_+^\dagger \end{pmatrix} \begin{pmatrix} \mathbf{0} & \Gamma_\mu \\ \tilde{\Gamma}_\mu & \mathbf{0} \end{pmatrix} \begin{pmatrix} z_+ \\ z_- \end{pmatrix}\right] \\ &= \mathrm{Re}\,(w_+^\dagger \tilde{\Gamma}_\mu z_+ + w_-^\dagger \Gamma_\mu z_-) = \mathrm{Re}\,\mathrm{tr}\left(z_+ w_+^\dagger \tilde{\Gamma}_\mu + z_- w_-^\dagger \Gamma_\mu\right) \\ &= \tfrac{1}{2}\left[\mathrm{Re}\,\mathrm{tr}\left(z_+ w_+^\dagger \tilde{\Gamma}_\mu + z_- w_-^\dagger \Gamma_\mu\right) + \mathrm{Re}\,\mathrm{tr}\left((z_+ w_+^\dagger \tilde{\Gamma}_\mu + z_- w_-^\dagger \Gamma_\mu)^\dagger\right)\right] \\ &= \tfrac{1}{2}\mathrm{Re}\,\mathrm{tr}\left([z_+ w_+^\dagger + w_+ z_+^\dagger]\tilde{\Gamma}_\mu + [z_- w_-^\dagger + w_- z_-^\dagger]\Gamma_\mu\right) \\ &= \tfrac{1}{2}\mathrm{Re}\,\mathrm{tr}\left([z_+ w_+^\dagger + w_+ z_+^\dagger]\tilde{\Gamma}_\mu + [z_- \widetilde{w_-^\dagger + w_-} z_-^\dagger]\tilde{\Gamma}_\mu\right). \end{aligned} \qquad (2.198)$$

So the vector covariant is formed of combinations of spinors of the same chirality. Since the hermitian matrix $\mathbf{Y}$ is completely determined by the components according to (2.194) and the terms in square brackets are hermitian, we can give a formula analogous to (2.135):

$$\begin{aligned} \not{y} &= \begin{pmatrix} 0 & \mathbf{Y} \\ \tilde{\mathbf{Y}} & 0 \end{pmatrix} := \gamma^k \mathrm{Re}\,\overline{w}\gamma_k z \\ &= \begin{pmatrix} \mathbf{0} & [z_+ w_+^\dagger + w_+ z_+^\dagger] + [z_- \widetilde{w_-^\dagger + w_-} z_-^\dagger] \\ [z_+ \widetilde{w_+^\dagger + w_+} z_+^\dagger] + [z_- w_-^\dagger + w_- z_-^\dagger] & \mathbf{0} \end{pmatrix}. \end{aligned} \qquad (2.199)$$

Proper Lorentz transformations are generated by pairs of timelike (resp. spacelike) unit vectors $u, v \in V$, i.e., $u_\mu u^\mu = \mp 1 = v_\mu v^\mu$. We choose $v = e_{-1}$ fixed

$$\begin{aligned} \not{x}' &= \not{u}\gamma_{-1}\not{x}\gamma_{-1}\not{u} = \begin{pmatrix} \mathbf{0} & \mathbf{UXU} \\ \tilde{\mathbf{U}}\tilde{\mathbf{X}}\tilde{\mathbf{U}} & \end{pmatrix}, \\ w' &= \not{u}\gamma_{-1} w = \begin{pmatrix} -\mathbf{U}w_+ \\ \tilde{\mathbf{U}}w_- \end{pmatrix}. \end{aligned} \qquad (2.200)$$

The correct transformation behavior of spinors and vectors is ensured by the Moufang identities as in the 8-dimensional case, since $\not{\psi}$ contains additional real parameters but only one full octonion. This form of proper Lorentz transformations makes the isomorphism $SL(2,\mathbb{O}) \cong SO(9,1)$ as Lie groups precise.

Since for $C := \gamma_{-1}\gamma_0\gamma_8 = -\epsilon \otimes \epsilon$

$$C\gamma_\mu = \gamma_\mu^* C \qquad (-1 \leq \mu \leq 8), \tag{2.201}$$

a "charge conjugation" operation is given by

$$w_C := C(w)^* = -\epsilon \otimes \epsilon\, w^* = \begin{pmatrix} \epsilon\, w_-^* \\ -\epsilon\, w_+^* \end{pmatrix}, \tag{2.202}$$

which must involve the opposite octonionic algebra as it was pointed in (2.150) and (2.151). This transition to the opposite algebra for spinors with opposite chirality may be useful in theories with $N > 1$ supersymmetry.

Of course, we may iterate the process of shrinking and extending of a representation with $\gamma_{9,1}$ as a starting point. We can shrink it to obtain representations of $Cl_0(9,1) \cong Cl(9,0) \cong Cl(1,8)$ and from there to $Cl_0(9,0) \cong Cl(0,8)$ and $Cl_0(1,8) \cong Cl(1,7)$. Also an extension to a representation of $Cl(10,2)$ is possible.

2.6. AN OCTONIONIC DESCRIPTION OF THE CHEVALLEY ALGEBRA AND TRIALITY

The triality automorphisms of the Chevalley algebra are well known and have been discussed in detail before [29,8,9], even in an octonionic formulation [30]. However, in our opinion, the following treatment based on the preparatory work of section 2.4 adds another unique and very transparent perspective with regard to this topic.

In the case of 8 euclidean dimensions we are in a special situation; the spaces of vectors, V, even spinors, S_0, and odd spinors, S_1, have the same dimension, namely 8. This allows the construction of the triality maps that interchange the transformation behavior of these three spaces. We define the Chevalley algebra $\mathcal{A} := V \oplus S_0 \oplus S_1$ to be the direct sum of these three spaces. This definition automatically provides a vector space structure for $\mathcal{A}$. Furthermore, $\mathcal{A}$ inherits an $SO(8)$-invariant bilinear form $\mathbf{B} = 2\,g \oplus 2\,A$ from the metric g on the vector space and the hermitian form A on $S = S_0 \oplus S_1$. (For notational convenience later on, we put in a factor of 2 in the definition of $\mathbf{B}$.) For $a = a_v \oplus a_0 \oplus a_1$, $b = b_v \oplus b_0 \oplus b_1 \in \mathcal{A}$, we obtain

$$\mathbf{B}(a,b) = 2\,g(a_v, b_v) + 2\,A(\begin{pmatrix} a_0 \\ a_1^* \end{pmatrix}, \begin{pmatrix} b_0 \\ b_1^* \end{pmatrix}) = 2\,\mathrm{Re}\,(a_v b_v^* + a_0^* b_0 + a_1 b_1^*), \tag{2.203}$$

where we used the parametrization of the spinor components introduced in section 2.4.5. (2.203) confirms that A decomposes and is a real symmetric bilinear form on the 16 real spinor components. The $SO(8)$-invariance of $\mathbf{B}$ is clear using the results of section 2.4.3. Furthermore, we observed in (2.143) that the expression

$$\mathbf{T}'(a) := \mathrm{Re}\,\overline{a_1} \not{a}_v^* a_0 = \mathrm{Re}\left[\begin{pmatrix} 0 & a_1 \end{pmatrix} \begin{pmatrix} 0 & a_v^* \\ a_v & 0 \end{pmatrix} \begin{pmatrix} a_0 \\ 0 \end{pmatrix}\right] = \mathrm{Re}\,a_1 a_v a_0 \tag{2.204}$$

is $SO(8)$-invariant. (Note that, we also redefined our basis of V by octonionic conjugation for symmetry reasons, which will become relevant below.) By polarization, we define a $SO(8)$-invariant symmetric trilinear form on $\mathcal{A}$, which we denote by $\mathbf{T}$:

$$\mathbf{T}(a,b,c) := \mathrm{Re}\,(a_1 b_v c_0 + a_1 c_v b_0 + b_1 a_v c_0 + b_1 c_v a_0 + c_1 a_v b_0 + c_1 b_v a_0) \quad (a,b,c \in \mathcal{A}). \tag{2.205}$$

The Chevalley product "$\circ_{\mathcal{A}}$" is then implicitly defined to satisfy the following condition connecting $\mathbf{B}$ and $\mathbf{T}$:

$$\mathbf{B}(a \circ_{\mathcal{A}} b, c) = \mathbf{T}(a,b,c) \qquad \forall\, a,b,c \in \mathcal{A}. \tag{2.206}$$

The Chevalley product is obviously symmetric and $SO(8)$ invariant.

In this setting the triality maps are just automorphisms of the Chevalley algebra, which interchange V, S_0, and S_1. But before we describe the triality maps, we will take advantage of the octonionic formalism and rewrite the bilinear and trilinear forms, $\mathbf{B}$ and $\mathbf{T}$, and the Chevalley product by representing elements of the Chevalley algebra by octonionic hermitian 3×3-matrices with vanishing diagonal elements,

$$a = \begin{pmatrix} 0 & a_v^* & a_0 \\ a_v & 0 & a_1^* \\ a_0^* & a_1 & 0 \end{pmatrix} = \begin{pmatrix} \not{a}_v^* & a_s \\ \overline{a_s} & 0 \end{pmatrix} \in \mathcal{A}, \tag{2.207}$$

where $a_s = \binom{a_0}{a_1^*} = a_o \oplus a_1 \in S$. Then the bilinear form $\mathbf{B}$ is given by

$$\begin{aligned} \mathbf{B}(a,b) &= \tfrac{1}{2}\mathrm{tr}\,(ab+ba) = \mathrm{tr}\,(a \circ b) \\ &= \tfrac{1}{2}\mathrm{tr}\left(\begin{pmatrix} 0 & a_v^* & a_0 \\ a_v & 0 & a_1^* \\ a_0^* & a_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & b_v^* & b_0 \\ b_v & 0 & b_1^* \\ b_0^* & b_1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & b_v^* & b_0 \\ b_v & 0 & b_1^* \\ b_0^* & b_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & a_v^* & a_0 \\ a_v & 0 & a_1^* \\ a_0^* & a_1 & 0 \end{pmatrix}\right) \\ &= \tfrac{1}{2}\mathrm{tr}\left(\begin{pmatrix} a_v^* b_v + a_0 b_0^* & a_0 b_1 & a_v^* b_1^* \\ a_1^* b_0^* & a_v b_v^* + a_1^* b_1 & a_v b_0 \\ a_1 b_v & a_0^* b_v^* & a_0^* b_0 + a_1 b_1^* \end{pmatrix}\right) \\ &\quad + \tfrac{1}{2}\mathrm{tr}\left(\begin{pmatrix} b_v^* a_v + b_0 a_0^* & b_0 a_1 & b_v^* a_1^* \\ b_1^* a_0^* & b_v a_v^* + b_1^* a_1 & b_v a_0 \\ b_1 a_v & b_0^* a_v^* & b_0^* a_0 + b_1 a_1^* \end{pmatrix}\right) \\ &= \tfrac{1}{2}[(a_v^* b_v + b_v^* a_v + a_v b_v^* + b_v a_v^*) + (a_0 b_0^* + b_0 a_0^* + a_0^* b_0 + b_0^* a_0) \\ &\qquad + (a_1^* b_1 + b_1^* a_1 + a_1 b_1^* + b_1 a_1^*)] \\ &= 2\,\mathrm{Re}\,(a_v b_v^* + a_0^* b_0 + a_1 b_1^*), \end{aligned} \tag{2.208}$$

where "$\circ$" denotes the symmetrized matrix product

$$a \circ b := \frac{1}{2}(ab + ba). \tag{2.209}$$

In fact, the symmetrized product is the Jordan product and the matrices that we are dealing with are a subset of the exceptional Jordan algebra of 3×3 octonionic hermitian matrices [10].

For the trilinear form $\mathbf{T}$ we find

$$\begin{aligned}\mathbf{T}(a, b, c) &= \operatorname{tr}((a \circ b) \circ c) \\ &= \tfrac{1}{4}[(a_0 b_1 c_v + c_v a_0 b_1 + b_1^* a_0^* c_v^* + c_v^* b_1^* a_0^*) + (b_0 a_1 c_v + c_v b_0 a_1 \\ &\quad + a_1^* b_0^* c_v^* + c_v^* a_1^* b_0^*) + (a_v b_0 c_1 + c_1 a_v b_0 + b_0^* a_v^* c_1^* + c_1^* b_0^* a_v^*) \\ &\quad + (b_v a_0 c_1 + c_1 b_v a_0 + a_0^* b_v^* c_1^* + c_1^* a_0^* b_v^*) + (a_1 b_v c_0 + c_0 a_1 b_v \\ &\quad + b_v^* a_1^* c_0^* + c_0^* b_v^* a_1^*) + (b_1 a_v c_0 + c_0 b_1 a_v + a_v^* b_1^* c_0^* + c_0^* a_v^* b_1^*)] \\ &= \operatorname{Re}(b_1 c_v a_0 + a_1 c_v b_0 + c_1 a_v b_0 + c_1 b_v a_0 + a_1 b_v c_0 + b_1 a_v c_0).\end{aligned} \tag{2.210}$$

It follows from (2.206), (2.208), and (2.210) that the Chevalley product "$\circ_{\mathcal{A}}$" is given by the off-diagonal elements of the symmetrized matrix product "$\circ$",

$$\operatorname{tr}((a \circ_{\mathcal{A}} b) \circ c) = \mathbf{B}(a \circ_{\mathcal{A}} b, c) = \mathbf{T}(a, b, c) = \operatorname{tr}((a \circ b) \circ c) \tag{2.211}$$

$$\Longrightarrow \qquad (a \circ_{\mathcal{A}} b) = (a \circ b)_{\mathcal{A}}, \tag{2.212}$$

where the subscript "$\mathcal{A}$" on a matrix denotes the matrix with erased diagonal elements, i.e.,

$$(a \circ b)_{\mathcal{A}} := \begin{pmatrix} 0 & a_0 b_1 + b_0 a_1 & a_v^* b_1^* + b_v^* a_1^* \\ a_1^* b_0^* + b_1^* a_0^* & 0 & a_v b_0 + b_v a_0 \\ a_1 b_v + b_1 a_v & a_0^* b_v^* + b_0^* a_v^* & 0 \end{pmatrix}. \tag{2.213}$$

(Note that only the off diagonal elements of $a \circ b$ contribute to the last term of (2.212)). Traditionally the Chevalley product is written in terms of Clifford products, which we combine into the 3×3-matrix

$$a \circ_{\mathcal{A}} b = \begin{pmatrix} \Gamma^k \overline{a_s} \Gamma_k b_s & \not a_v^* b_s + \not b_v^* a_s \\ \overline{a_s} \not b_v^* + \overline{b_s} \not a_v^* & 0 \end{pmatrix} \tag{2.214}$$

What we have done is to utilize the Jordan product and project onto the Chevalley algebra. Since both **B** and **T** are expressed entirely in terms of the Jordan product, automorphisms of the Jordan product, that map the Chevalley algebra onto itself, will also be automorphisms of the Chevalley algebra. We have already encountered one such automorphism, namely the orthogonal transformation corresponding to a generator $p_v \in V$ with $|p_v|^2 = 1$, which is written in matrix form

$$\tau_{p_v}(a) := \begin{pmatrix} 0 & p_v^* & 0 \\ p_v & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & a_v^* & a_0 \\ a_v & 0 & a_1^* \\ a_0^* & a_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & p_v^* & 0 \\ p_v & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \not p_v^* \not a_v^* \not p_v^* & \not p_v^* a_s \\ \overline{a_s} \not p_v^* & 0 \end{pmatrix} . \tag{2.215}$$

This first triality map combines the vector action and spinor action of the Clifford group (see section 2.3.2). The action of the generator p_v is a reflection at a hyperplane orthogonal to p_v combined with an inversion of the whole space. This transformation is an improper rotation and interchanges even and odd spinors:

$$\begin{aligned} \tau_{p_v}(a_v) &= p_v a_v^* p_v \quad \in V\,, \\ \tau_{p_v}(a_0) &= (p_v a_0)^* \quad \in S_1\,, \\ \tau_{p_v}(a_1) &= (a_1 p_v)^* \quad \in S_0\,. \end{aligned} \tag{2.216}$$

Using the Moufang identities, it is easy to check that τ_{p_v} is indeed an automorphism of $\mathcal{A}$ of order 2, i.e., $\tau_{p_v}^2 = 1$. Composing an even number of maps τ_{p_v} with different parameters p_v, we generate the simple orthogonal group $SO(8)$ as is seen in (2.141) and (2.142). From the form of (2.215), it is obvious that there are two more families of automorphisms of $\mathcal{A}$ of order 2, parametrized by an even spinor variable p_0 and an odd spinor variable p_1 with $|p_0|^2 = 1 = |p_1|^2$:

$$\tau_{p_0}(a) := \begin{pmatrix} 0 & 0 & p_0 \\ 0 & 1 & 0 \\ p_0^* & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & a_v^* & a_0 \\ a_v & 0 & a_1^* \\ a_0^* & a_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & p_0 \\ 0 & 1 & 0 \\ p_0^* & 0 & 0 \end{pmatrix} \tag{2.217}$$

and

$$\tau_{p_1}(a) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & p_1^* \\ 0 & p_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & a_v^* & a_0 \\ a_v & 0 & a_1^* \\ a_0^* & a_1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & p_1^* \\ 0 & p_1 & 0 \end{pmatrix} . \tag{2.218}$$

For these two families of maps, the matrix formalism shows the clear parallel structure to the maps τ_{p_v}. Traditionally expressions in terms of both Clifford products and the spinor bilinear form are used for the maps τ_{p_0} and τ_{p_1}, which obscures this symmetry, because in τ_{p_v} only Clifford products are used. These two families preserve one of the spinor spaces and interchange the other one with V:

$$\begin{aligned} \tau_{p_0}(a_v) &= (a_v p_0)^* \ \in S_1 , \\ \tau_{p_0}(a_0) &= p_0 a_0^* p_0 \ \in S_0 , \\ \tau_{p_0}(a_1) &= (p_0 a_1)^* \ \in V , \end{aligned} \tag{2.219}$$

and

$$\begin{aligned} \tau_{p_1}(a_v) &= (p_1 a_v)^* \ \in S_0 , \\ \tau_{p_1}(a_0) &= (a_0 p_1)^* \ \in V , \\ \tau_{p_1}(a_1) &= p_1 a_1^* p_1 \ \in S_1 , \end{aligned} \tag{2.220}$$

By combining two triality maps with the same octonionic parameter $p_v = p = p_0$ from different families, we obtain a automorphisms Ξ_p of order 3:

$$\begin{aligned} & \Xi_p = \tau_{p_v=p} \circ \tau_{p_0=p} \\ \Longrightarrow \quad \Xi_p(a) = & \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & p^* \\ p & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & a_v^* & a_0 \\ a_v & 0 & a_1^* \\ a_0^* & a_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & p^* \\ 1 & 0 & 0 \\ 0 & p & 0 \end{pmatrix} \qquad (a \in \mathcal{A}), \end{aligned} \tag{2.221}$$

hence

$$\begin{aligned} \Xi_p(a_v) &= p^* a_v \quad \in S_0\,, \\ \Xi_p(a_0) &= p a_0 p \quad \in S_1\,, \\ \Xi_p(a_1) &= p^* a_1 \quad \in V\,. \end{aligned} \tag{2.222}$$

As is seen from their matrix forms, $\tau_{p_v=p}$ and Ξ_p generate Σ_3, the permutation group on three letters. (In particular for $p = 1$, this is easy to verify.) We observed before that the maps τ_{p_v} generate $O(8)$, so that the triality maps, we have found so far, have a group structure isomorphic to $\Sigma_3 \times SO(8)$. It is known (see [8]) that this is the full automorphism group of the Chevalley algebra, which is also the automorphism group of $SO(8)$. This concludes our demonstration of triality.

2.7. FINITE VS. INFINITESIMAL GENERATORS

In this article we characterize orthogonal groups in terms of a set of finite generators. This approach is not as widely used as the description in terms of infinitesimal generators, i.e., the Lie algebra of the group. In this section we compare the two approaches.

If we want to compare two Lie groups given by infinitesimal generators we know how to proceed [31]. We determine their Lie algebra by working out the commutators of the generators. We then determine their structure constants and identify the Lie algebra. For semi-simple Lie algebras the Cartan-Weyl normalization provides a unique identification. We may also use a Lie algebra homomorphism and determine its image and kernel to relate the two groups in question. Whether the homomorphism is surjective and injective can often be determined by counting the dimension of the Lie algebras involved. Having identified the Lie algebra we have full knowledge of the local structure of the Lie group. From this information we can construct the simply connected universal covering group, which has this local structure. However, the Lie group we are trying to characterize may be neither connected nor simply connected. So in order to compare two groups we need to have some global information about them in addition to the infinitesimal generators.

In section 2.3.2 we compared two groups given by finite generators, namely the orthogonal group generated by reflections on hyperplanes and the Clifford group generated by non-null vectors of the Clifford algebra. The relationship was established considering a group homomorphism. The homomorphism is surjective if the generators lie in the image. This is the analogue to counting the dimension of the Lie algebras. Determining the kernel, which has to be a normal subgroup, completes the comparison. The advantage of finite generators is the global information that they carry. Having found an isomorphism based on the finite generators, we know that the groups have the same global structure.

Even though the two descriptions have different features, they are closely related. The exponential map provides a means to parametrize a neighborhood of the identity element of the group. This coordinate chart can be translated by a finite element in this neighborhood, hence we can construct an atlas of the component of the group that is connected to the identity. Actually, we need information about the global structure to patch the charts together correctly. For an additional component of the group that is not connected to the identity, we may use the same atlas, since the components are diffeomorphic.

The finite generators that determine the groups considered in this article are elements of a topological manifold of dimension less than the dimension of the group. For example, the octonions that generate $SO(8)$ (2.141) are elements of the octonionic unit sphere, S^7. Translating a disk centered at a point $p \in S^7$ by $p^{-1} \in S^7$, we obtain a submanifold of the group containing the identity. (A generating set of a group is always assumed to contain inverses of every element.) This submanifold is of lower dimension than the Lie group, so its tangent space at the identity is only a linear subspace of the Lie algebra. In most of our examples it is sufficient to consider the translation of a sufficient number of disks contained in the generating set to obtain linear subspaces that span the Lie algebra. Otherwise the process continues by taking products of elements of two disks around p_1 and p_2 in the generating set and translating these products by $(p_1 p_2)^{-1}$ to the identity. An example of this latter construction is the S^6 generating $SO(8)$ described in [4]. In this way infinitesimal generators can be found starting from finite ones.

There is also a formal construction of the entire group; namely, the group is given by the set of equivalence classes of finite sequences of generators. The group product of two elements $[g_1], [g_2]$ is just the class of the juxtaposition $[g_1 g_2]$ of two representatives. For the octonionic description we need to do this decomposition into generators to find spinor and vector transformations that are consistent. For example, if a vector given by $x \in \mathbb{O}$ transforms by $x \mapsto uxu^*$, which is an $SO(8)$ transformation, we need to re-express uxu^* as $v_1(v_2(\ldots(v_k x v_k)\ldots)v_2)v_1$ with $|v_1|^2 = |v_2|^2 = \cdots = |v_k|^2$ in order to determine the corresponding spinor transformation $w \mapsto v_1(v_2(\ldots(v_k w)\ldots))$. In general, octonionic transformations, because of their non-associativity, involve this nesting of multiplications. Therefore the octonionic description of Lie groups in terms of generators is the natural one. Octonionic descriptions of Lie algebras, which are also possible, have the disadvantage that the exponential map no longer works because of the non-associativity. So this avenue does not provide a construction of finite group elements.

2.8. CONCLUSION

We have demonstrated that the abstract octonionic algebra is a suitable structure to represent Clifford algebras in certain dimensions. We obtained most of our results from the basic property of composition algebras, which is the norm compatibility of multiplication, and its consequence alternativity. The alternative property, in particular in the form of the Moufang identities, was found to be responsible for ensuring the correct transformation behavior of octonionic spinors and for ensuring the consistency of the representation in terms of left multiplication by octonionic matrices. The choice of a multiplication rule for the octonions, in particular, the modified "X-product", was found to be related to coordinate transformations or a change of basis of the spinor space. The opposite octonionic algebra was shown to be connected to an analogue of the charge conjugate representation. The Clifford group and its action on vectors and spinors led to octonionic representations of orthogonal groups in corresponding dimensions. The natural octonionic description of these groups is in terms of generating sets of the Lie group rather than in terms of generators of the Lie algebra. This is due to the nested structure which is necessary to accommodate the non-associativity of the octonions.

The usefulness of this tool of octonionic representations was evident in the presentation of the triality automorphisms of the Chevalley algebra. This presentation unequivocally showed that the spaces of vectors and even and odd spinors are interchangeable in this case. We expect that a similar, fully octonionic treatment of supersymmetrical theories will make their symmetries more transparent. In fact, we have successfully applied the methods of this article to the CBS-superparticle [32]. We hope to be able to find a parallel treatment of the Green-Schwarz superstring.

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3. FINITE LORENTZ TRANSFORMATIONS, AUTOMORPHISMS, AND DIVISION ALGEBRAS

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We give an explicit algebraic description of finite Lorentz transformations of vectors in 10-dimensional Minkowski space by means of a parameterization in terms of the octonions. The possible utility of these results for superstring theory is mentioned. Along the way we describe automorphisms of the two highest dimensional normed division algebras, namely the quaternions and the octonions, in terms of conjugation maps. We use similar techniques to define

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$SO(3)$ and $SO(7)$ via conjugation, $SO(4)$ via symmetric multiplication, and $SO(8)$ via both symmetric multiplication and one-sided multiplication. The non-commutativity and non-associativity of these division algebras plays a crucial role in our constructions.

3.1. INTRODUCTION

Recent research by several groups [1] on the $(9,1)$ dimensional[1] superstring has shown that a parameterization in terms of octonions is natural and may help to illuminate the symmetries of the theory. In particular, an isomorphism between $SO(9,1)$ and $SL(2,\mathbb{O})$ can be used to write the $(9,1)$ vector made up of the bosonic coordinates of the superstring as a 2×2 dimensional hermitian matrix with octonionic entries in the same way that the standard isomorphism between $SO(3,1)$ and $SL(2,\mathbb{C})$ is used to write a $(3,1)$ vector as a 2×2 dimensional hermitian matrix with complex entries. But what exactly is meant by $SL(2,\mathbb{O})$? The infinitesimal version of $SL(2,\mathbb{O})$ has been known for some time [2]. However, since the octonions are not associative, it is not possible to "integrate" the infinitesimal transformations to obtain a finite transformation in the usual way. In this paper, we show how to get around this problem and give an explicit algebraic description of finite transformations in $SL(2,\mathbb{O})$. Along the way, we also develop explicit octonionic characterizations of the finite transformations of a number of other interesting groups, especially G_2, $SO(7)$, and $SO(8)$.

In Section 2 we present some basic information about division algebras and introduce our notation. This section may be safely omitted by the reader who is already familiar with division algebras. In Section 3 we give an explicit algebraic description of finite elements of $SO(3)$ and $SO(7)$. ($SO(3)\cong Aut(\mathbb{H})$ is the group of continuous proper automorphisms of the quaternions.) We also find a simple

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[1]For notational convenience we use the symbol (m,l) to denote the total dimension of Minkowski space, where m is the number of spatial dimensions and l is the number of timelike dimensions.

restriction of $SO(7)$ which gives a construction of the continuous proper automorphisms of the octonions $G_2 \cong Aut(\mathbb{O})$. Then in Section 4 we find a related algebraic description of $SO(4)$ and two descriptions of $SO(8)$. We use these results in Section 5 to construct **finite** Lorentz transformations of vectors in $(5,1)$ and $(9,1)$ dimensions. Section 6 summarizes our conclusions and discusses how our work relates to the work of others.

3.2. DIVISION ALGEBRA BASICS

In this section we introduce the basic definitions and properties of the normed division algebras. We take an intuitive approach in order to make a first encounter accessible. For a more rigorous mathematical treatment see, for example, [3].

According to a theorem by Hurwitz [4], there are only four algebras over the reals, called normed division algebras, with the property that their norm is compatible with multiplication. These are the reals $\mathbb{R}$, the complexes $\mathbb{C}$, the quaternions $\mathbb{H}$, and the octonions $\mathbb{O}$; which we denote by $\mathbb{K}_n$, where $n = 1, 2, 4, 8$ is their respective dimension as vector spaces over the reals.

First we need to define these algebras. An element p of $\mathbb{K}_n$ is written[2] $p = p^i e_i$ for $p^i \in \mathbb{R}$, where $i = 1, \ldots, n$. The e_i's can be identified with an orthonormal basis in $\mathbb{R}^n$, but they also carry the information which determines the algebraic structure of $\mathbb{K}_n$. Addition on $\mathbb{K}_n$ is just addition of vectors in $\mathbb{R}^n$:

$$p + q = (p^i e_i) + (q^i e_i) = (p^i + q^i) e_i \tag{3.1}$$

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[2]Throughout this paper summation over repeated indices is implied unless otherwise noted.

and is therefore both commutative and associative. Multiplication is described by the tensor Λ. (Λ must be defined so as to contain the structural information necessary to yield norm compatibility. We discuss the detailed properties of Λ below.)

$$pq = (p^j e_j)(q^k e_k) = (\Lambda^i{}_{jk} p^j q^k) e_i \tag{3.2}$$

where $\Lambda^i{}_{jk} \in \mathbb{R}$ for $i, j, k = 1, \ldots, n$. We see that multiplication is bilinear and distributive, i.e. determined by the products of the basis vectors, but it is not necessarily commutative nor even associative.

We write the multiplicative identity in $\mathbb{K}_n$ as $e_1 = 1$ and call it the real unit.[3] Due to the linearity of (3.2), $\mathbb{R}e_1$ is an embedding of $\mathbb{R}$ in $\mathbb{K}_n$ and multiplication with an element of $\mathbb{R} \cong \mathbb{R}e_1$ is commutative. The other basis vectors satisfy $e_i e_i = e_i^2 = -1 = -e_1$ for $i = 2, \ldots, n$ and we call them imaginary basis units. The imaginary basis units anticommute with each other, i.e. $e_i e_j = -e_j e_i$ for $i \neq j$ and the product of two imaginary basis units yields another, i.e. $e_i e_j = \pm e_k$ for some k.

In the familiar way, we have $\{e_1 = 1\}$ for $\mathbb{R}$ and $\{e_1 = 1, e_2 = i\}$ for $\mathbb{C}$. For $\mathbb{H}$ we have $\{e_1 = 1, e_2, e_3, e_4 = e_2 e_3\}$. Because there is more than one imaginary basis unit, multiplication on $\mathbb{H}$ is not commutative, but it is still associative. The rest of the multiplication table follows from associativity. We can visualize multiplication in $\mathbb{H}$ by an oriented circle[4]; see Fig. 3.1. The product of two imaginary basis units, represented by nodes on the circle, is the imaginary basis unit represented by the third node on the line connecting them if the product is taken in the order

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[3]In most references the identity is denoted by e_0 or i_0, and indices run from 0 through $n-1$. For later notational convenience our indices run from 1 through n.

[4]In the figures and occasionally in the text, we will drop the e from the notation for a basis unit and refer to it just by its number, i.e. $e_2 \equiv 2$ and $e_i \equiv i$.

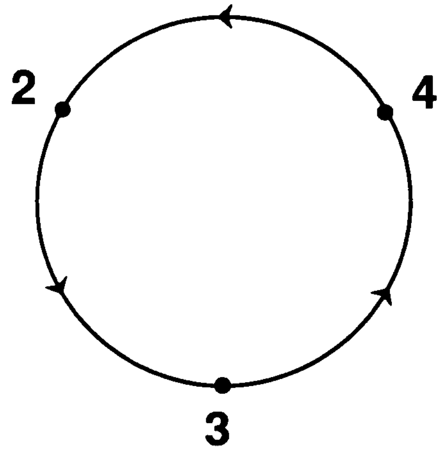

FIG. 3.1. A schematic representation of our choice for the quaternionic multiplication table.

given by the orientation of the circle, otherwise there is a minus sign in the result. Multiplication of the imaginary basis units in $\mathbb{H}$ is reminiscent of the vector product in $\mathbb{R}^3$: $\vec{\imath} \times \vec{\jmath} = \vec{k} = -\vec{\jmath} \times \vec{\imath}$. Because of this, e_2, e_3, e_4 are often denoted $\imath, \jmath, k$.

For $\mathbb{O}$ the multiplication table is most transparent when written as a triangle; see Fig. 3.2.

The product of two imaginary basis units is determined as before by following the oriented line connecting the corresponding nodes, where each line on the triangle is to be interpreted as a circle by connecting the ends. Moving opposite to the orientation of the line again contributes a minus sign, e.g. $e_3e_4 = e_2$ or $e_8e_6 = -e_3$. In general, multiplication in $\mathbb{O}$ is not associative, but e_1 and any triple of imaginary basis units lying on a single line span a 4-dimensional vector space isomorphic to $\mathbb{H}$. Therefore products of octonions from within such a subspace are associative. Products of triples of imaginary basis units not lying on a single line are precisely anti-associative so switching parentheses results in a change of sign. For example, $e_2(e_3e_4) = e_2(e_2) = -1 = (e_4)e_4 = (e_2e_3)e_4$, but $e_2(e_3e_5) = e_2(-e_7) = -e_8 = -(e_4)e_5 = -(e_2e_3)e_5$.

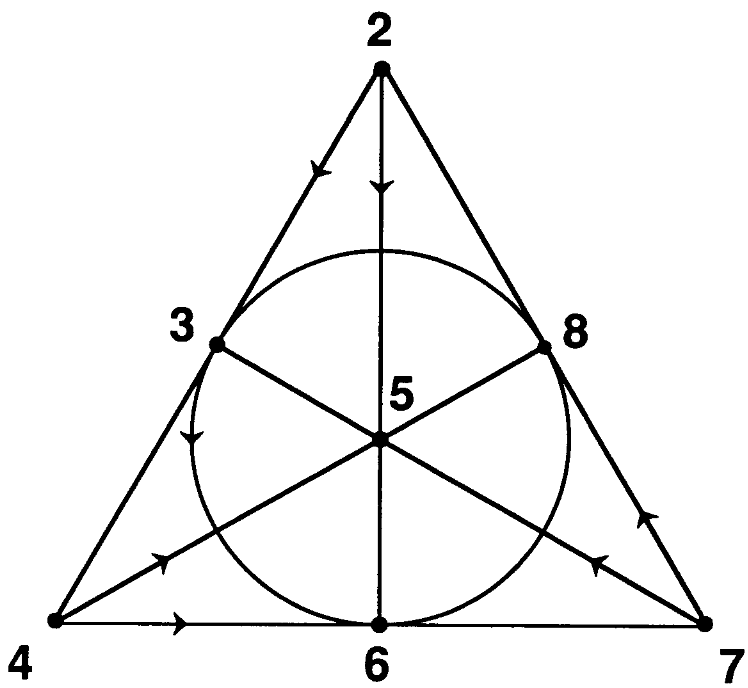

FIG. 3.2. A schematic representation of our choice for the octonionic multiplication table.

To describe the results of switching parentheses, it is useful to define the associator $[p,q,r] := p(qr) - (pq)r$ of three octonions p,q,r. The associator is totally antisymmetric in its arguments. From the antisymmetry of the associator we see that the octonions have a weak form of associativity, called alternativity, i.e. if the imaginary parts of any two of p,q,r point in the same direction in $\mathbb{R}^7$, the associator is zero. In particular, $[p,q,p] = 0$. As a consequence of alternativity, some products involving four factors have special associativity properties given by the Moufang [5] identities:

$$\begin{aligned} q\,(p\,(qx)) &= (qpq)\,x \\ ((xq)\,p)\,q &= x\,(qpq) \qquad\qquad \forall p,q,x,y \in \mathbb{K}_n \qquad (3.3) \\ q\,(xy)\,q &= (qx)\,(yq) \end{aligned}$$

As in the familiar case of the complex numbers, complex conjugation is accomplished by changing the sign of the components of the imaginary basis units, i.e. the complex conjugate of $p := p^i e_i$ is given by

$$p^* = \mathrm{Bar}(p) := p^1 e_1 - \sum_{i=2}^{n} p^i e_i \tag{3.4}$$

We define the real and imaginary parts[5] of p via

$$\mathrm{Re}\,p := \frac{1}{2}(p + p^*) \qquad \text{and} \qquad \mathrm{Im}\,p := \frac{1}{2}(p - p^*) \tag{3.5}$$

The complex conjugate of a product is the product of the complex conjugates in the opposite order:

$$(pq)^* = q^*\,p^*, \qquad \forall\, p,q \in \mathbb{K}_n \tag{3.6}$$

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[5]Note that $\mathrm{Im}\,p$ as we define it is not real. For $\mathbf{H}$ and $\mathbf{O}$ which have more than one imaginary direction, this definition is more convenient than the usual one.

The inner product on $\mathbb{K}_n$ is just the Euclidian one inherited from $\mathbb{R}^n$:

$$\langle p, q\rangle = \sum_{i=1}^{n} p^i q^i \tag{3.7}$$

which can be written in terms of complex conjugation via

$$\langle p, q\rangle = \frac{1}{2}(p\,q^* + q\,p^*) = \frac{1}{2}(q^*\,p + p^*\,q) = \mathrm{Re}(p\,q^*) \tag{3.8}$$

In this language, an imaginary unit is any vector which is orthogonal to the real unit and has norm 1. Two imaginary units which anticommute are orthogonal. This geometric picture relating orthogonality to anticommutativity is often helpful, but it lacks the notion of associativity.

The inner product, (3.7) and (3.8), induces a norm on $\mathbb{K}_n$ given by

$$|p| = |p^i e_i| = \sqrt{\sum_{i=1}^{n}(p^i)^2} = \sqrt{p\,p^*} \tag{3.9}$$

It can be shown that the norm is compatible with multiplication in $\mathbb{K}_n$:

$$|pq| = |p||q| \tag{3.10}$$

In the case of the octonions, (3.10) is known as the eight squares theorem, because a product of two sums, each of which consists of eight squares, is written as a sum of eight squares. Norm compatibility (3.10) and the relation of the norm to complex conjugation (3.9) are essential for a normed division algebra, since they allow division. For $p \neq 0$, the inverse of p is given by

$$p^{-1} = \frac{p^*}{|p|^2} \tag{3.11}$$

An element $p \in \mathbb{K}_n$ can be written in exponential form just as in the complex case:

$$p = N\exp(\theta\,\hat{r}) = N\,(\cos\theta + \sin\theta\,\hat{r}) \tag{3.12}$$

where $N = |p| \in \mathbb{R}$, $\theta \in [0, 2\pi)$ is given implicitly by $\mathrm{Re}\, p = N\cos\theta$, and $\hat{r}$ is an imaginary unit[6] given implicitly by $\mathrm{Im}\, p = N \sin\theta\, \hat{r}$. For the special case $N = 1$ we will sometimes denote p by the ordered pair

$$p = (\hat{r}, \theta) \tag{3.13}$$

What are the mth roots of $p = N\exp(\theta\, \hat{r}) \in \mathbb{K}_n$? If p is not a real number, then in the plane determined by e_1 and $\hat{r}$ the calculation reduces to the complex case, i.e. there are precisely m mth roots given by

$$p^{\frac{1}{m}} = N^{\frac{1}{m}} \exp\left(\frac{\theta + 2\pi l}{m}\hat{r}\right) \tag{3.14}$$

where $m \geq 2$ is a positive integer, $l < m$ is a non-negative integer, and $N^{\frac{1}{m}}$ is the positive, real mth root of the positive, real number N. However for K_4 and K_8, if $p \in \mathbb{K}_n$ is a real number the situation is different. If p is real, it does not determine a unique direction $\hat{r}$ in the pure imaginary space of $\mathbb{K}_n$. Therefore (3.14) is no longer well-defined (unless, of course, the root is real). Indeed, if $p_{\pm}{}^{\frac{1}{m}} = N^{\frac{1}{m}} \exp(\pm\frac{\theta+2\pi l}{m} e_2)$, for fixed l, are a complex conjugate pair of roots of p lying in $\mathbb{C}$, then $N^{\frac{1}{m}} \exp(\frac{\theta+2\pi l}{m}\hat{r})$ is also a root for any $\hat{r}$. We see that the roots of p, which form complex conjugate pairs in $\mathbb{C}$, in $\mathbb{K}_n$ form an S^{n-2} subspace of $\mathbb{R}^n$. Throughout this paper, whenever we refer to the root of an element of $\mathbb{K}_n$, we will mean any of these roots, so long as all of the roots of that element in a given equation are taken to be the same.

In the discussion so far we assumed that the basis $e_1, \ldots, e_n$ was given. But what happens if we change basis in $\mathbb{K}_n$? Any linear transformation would preserve the vector space structure of $\mathbb{K}_n$, but the structure tensor Λ would transform according to the tensor transformation rules. In order to preserve the multiplicative

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[6]We will use hats (e.g. $\hat{r}$) to denote purely imaginary units.

structure, i.e. to get the same multiplication rules and the same formulas for complex conjugation and norm, we would need for the transformation to be an automorphism of $\mathbb{K}_n$. Any such transformation yields a basis of the following form: (a) e_1 is the multiplicative identity in $\mathbb{K}_n$ and must be fixed by the transformation. For $\mathbb{R}$, $\{e_1\}$ is the basis. (b) e_2 can be any imaginary unit, i.e. anything in $\mathbb{K}_n$ which squares to -1. For $\mathbb{C}$ there is only one choice (up to sign), so the basis in this case is now complete. (c) e_3 can be any imaginary unit which anticommutes with e_2. Then e_4, the third unit in the associative triple, is determined by the multiplication table, i.e. $e_4 = e_2 e_3$. Now we have a basis for $\mathbb{H}$. (d) For $\mathbb{O}$ we still need to pick another imaginary unit, e_5, which anticommutes with all of e_2, e_3, and e_4. The remaining units are then determined by the triangle.

The procedure above provides a convenient simplification for calculations which involve up to three arbitrary octonions x, y, z. Without loss of generality, we may assume that $x = x^1 e_1 + x^2 e_2$, $y = y^1 e_1 + y^2 e_2 + y^3 e_3$, and $z = z^1 e_1 + z^2 e_2 + z^3 e_3 + z^4 e_4 + z^5 e_5$. In particular, any calculation involving only one arbitrary octonion reduces to the complex case and any involving only two arbitrary octonions reduces to the quaternionic case. In a calculation involving three arbitrary octonions, it may be assumed that only one component of one of them lies outside a single associative triple. Only the fourth arbitrary octonion in a calculation cannot be chosen to have some vanishing components. These simplifications can be especially useful when combined with computer algebra techniques.

The multiplication rules which we have chosen are not unique, but all other choices amount to renumberings of the circle or triangle, including those which switch signs (nodes may be relabeled $\pm 2, \ldots, \pm 8$). Even some of these turn out to be equivalent to the original triangle. The seven points of the triangle can be identified with the projective plane over the field with two elements, so the possible renumberings of the imaginary basis units correspond to transformations of this plane. For future reference we give the form of Λ corresponding to our choice of multiplication rules in Appendix A.

3.3. $SO(n-1)$ AND AUTOMORPHISMS

A proper automorphism ϕ of $\mathbb{K}_n$ satisfies

$$\phi(x+y) = \phi(x) + \phi(y) \tag{3.15}$$

$$\phi(xy) = \phi(x)\phi(y) \qquad \text{(proper)} \tag{3.16}$$

$\forall\, x, y \in \mathbb{K}_n$, whereas for an improper or anti-automorphism the order of the factors in (3.16) is reversed:

$$\phi(xy) = \phi(y)\phi(x) \qquad \text{(improper)} \tag{3.17}$$

From (3.6) and the non-commutativity of quaternionic and octonionic multiplication, we see that complex conjugation is an example of an improper automorphism for $n = 4, 8$.

Throughout the rest of this paper we will restrict ourselves to the set of continuous proper automorphisms, $Aut(\mathbb{K}_n)$.[7] Then (3.15), (3.16), and continuity are sufficient to show that ϕ is a linear transformation on $\mathbb{K}_n$. As such, ϕ can be expressed by the action of a real matrix $A^i{}_j$ acting on the components x^j (for $j = 1, \ldots, n$) of x viewed as a vector in $\mathbb{R}^n$:

$$\phi : \mathbb{K}_n \to \mathbb{K}_n \text{ linear} \iff \phi(x) = A^i{}_j x^j e_i \tag{3.18}$$

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[7]All of the continuous automorphisms of $\mathbb{H}$ or $\mathbb{O}$, including the improper ones which change the order of the multiplication, can be obtained by taking the direct product of $Aut(\mathbb{H})$ or $Aut(\mathbb{O})$ with the group $\{1, \mathrm{Bar}\}$.

Combining this form of ϕ with the condition (3.16) and using the multiplication rule (3.2) we obtain the following equation for the $A^i{}_j$'s:

$$A^i{}_l \Lambda^l{}_{jk} = A^l{}_j \Lambda^i{}_{lm} A^m{}_k \tag{3.19}$$

This equation defines the Lie group of automorphisms in terms of $n \times n$ matrices and the structure constants of $\mathbb{K}_n$.

The formulation which we have just described is the usual one for Lie groups, but it does not take advantage of the special algebraic structure of $\mathbb{K}_n$. The approach which we prefer to take in this paper is to find algebraic operations on $\mathbb{K}_n$ which yield maps that satisfy (3.15–(3.16) without resorting to the matrix description. The algebraic operations which we will find turn out to have many interesting properties.

Motivated by the structure of inner automorphism on division rings, let us consider conjugation maps ϕ_q on $\mathbb{K}_n = \mathbb{H}, \mathbb{O}$ $(n = 4, 8)$ for $q \in \mathbb{K}_n{}^* = \mathbb{K}_n - \{0\}$:

$$\begin{aligned} \phi_q : \mathbb{K}_n &\to \mathbb{K}_n \\ x &\mapsto qxq^{-1} \end{aligned} \tag{3.20}$$

These maps are well-defined even for $\mathbb{K}_8 = \mathbb{O}$ since the associator $[q, x, q^{-1}]$ vanishes. (This vanishing associator also implies that $(\phi_q)^{-1} = \phi_{q^{-1}}$ and $(\phi_q)^2 = \phi_{q^2}$ for both $\mathbb{H}$ and $\mathbb{O}$). The maps (3.20) satisfy (3.15) and fix the real part of x.

We see from (3.20) that a rescaling of q does not effect the transformation, so without loss of generality we may divide out the multiplicative center, $\mathbb{R}^* = \mathbb{R} - \{0\}$, and consider only q's of unit norm, i.e. $q = (\hat{r}, \theta)$.[8] Notice that now $q^{-1} = q^*$. Thus we have a map Φ which takes $\{q \in \mathbb{K}_n : |q| = 1\} \cong \mathbb{K}_n{}^*/\mathbb{R}^* \cong S^{n-1}$ to $\{\phi_q\}$, where ϕ_q is a linear transformation on $\mathbb{K}_n$:

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[8]We could also identify antipodal points on the unit sphere (S^{n-1}), since $\phi_q = \phi_{-q}$.

$$\Phi : \{q \in \mathbb{K}_n \,:\, |q| = 1\} \to L(\mathbb{K}_n, \mathbb{K}_n) \tag{3.21}$$

$$q \mapsto \phi_q = \phi_{(\hat{r},\theta)} : \mathbb{K}_n \to \mathbb{K}_n$$

$$x \mapsto qxq^* = \exp(\theta\,\hat{r})\,x\exp(-\theta\,\hat{r})$$

We see from (3.10) that ϕ_q is an isometry:

$$|\phi_q(x)| = |q||x||q^*| = |x| \tag{3.22}$$

In particular it leaves the norm of the imaginary part invariant so the associated $n \times n$ matrix A_q (which is defined by: $\phi_q(x) = (A_q)^i{}_j x^j e_i$) is orthogonal and splits into a trivial 1×1 block for the real part and an $(n-1) \times (n-1)$ block R_q which lies in $SO(n-1)$. The determinant of A_q is positive, because $\phi_q = (\phi_{\sqrt{q}})^2$ (equivalently $A_q = (A_{\sqrt{q}})^2$).

Now we will study the structure of $\Phi(S^{n-1})$ by looking at generic examples of maps ϕ_q.

3.3.1. Quaternions and $SO(3)$

For $\mathbb{K}_1 = \mathbb{R}$ and $\mathbb{K}_2 = \mathbb{C}$, multiplication is commutative and the conjugation maps (3.20) are trivial. Therefore let us examine the first nontrivial case, $\mathbb{K}_4 = \mathbb{H}$. If we consider, for example, $\hat{r} = e_2$, we get

$$\begin{aligned} &\exp(\theta\, e_2)\, x \exp(-\theta\, e_2) = \\ &x^1 e_1 + x^2 e_2 + (\cos 2\theta\; x^3 - \sin 2\theta\; x^4) e_3 + (\sin 2\theta\; x^3 + \cos 2\theta\; x^4) e_4 \end{aligned} \tag{3.23}$$

$$\iff A_{(e_2,\theta)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos 2\theta & -\sin 2\theta \\ 0 & 0 & \sin 2\theta & \cos 2\theta \end{bmatrix} \quad \text{so} \quad R_{(e_2,\theta)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta & -\sin 2\theta \\ 0 & \sin 2\theta & \cos 2\theta \end{bmatrix} \tag{3.24}$$

This is just a rotation of the imaginary part of x around e_2 by an angle of 2θ, i.e. it is a rotation in the 3-4 plane. Similarly, we see that ϕ_q with $q = \exp(\theta\,\hat{r})$, for any imaginary unit $\hat{r}$, is a rotation of the imaginary part of x around $\hat{r}$ by an angle of 2θ . Thus Φ is the universal covering map, mapping S^3 onto $SO(3) \cong Aut(\mathbb{H})$. Since multiplication in $\mathbb{H}$ is associative, composition of maps is given by multiplication in $\mathbb{H}$, i.e. $\phi_p \circ \phi_q = \phi_{pq}$ (equivalently $A_p A_q = A_{pq}$), $\forall\, p, q \in \mathbb{H}$ with $|p| = |q| = 1$. Therefore, Φ is also a group homomorphism.[9]

We have just parameterized rotations in the 3-dimensional purely imaginary subspace of the quaternions by fixing an axis of rotation and then specifying the value of a continuous parameter, the angle θ, which describes the amount of the rotation around that axis in the unique plane orthogonal to that axis. We call this parameterization the axis-angle form. But in dimension greater than 3, there is no unique plane orthogonal to a given axis. Therefore in the octonionic case it will not be sufficient to specify a rotation axis and an angle of rotation. Instead, we will parameterize rotations in another way, which we first describe here for the quaternionic case.

To accomplish a given elementary rotation (a rotation which takes place in a single coordinate plane), we use a composition of two particular axis-angle rotations, which we call flips because they are both rotations by the same constant angle π. The angle θ between the axes of the two flips then takes on the role of

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[9] One application of this homomorphism is a quick derivation of the expression for the composition of two rotations given in terms of axes and angles of rotation. If $p = \exp(\theta\,\hat{r})$ and $q = \exp(\eta\,\hat{s})$, then $pq = \exp(\zeta\,\hat{t})$ where $\hat{t} = \mathrm{Im}\,(pq)/|\mathrm{Im}\,(pq)|$ and $\cos\zeta = \mathrm{Re}\,(pq)$. So a 2η rotation around $\hat{s}$ followed by a 2θ rotation around $\hat{r}$ is the same as a 2ζ rotation around $\hat{t}$.

a continuously changing parameter which describes the magnitude of the combined rotation. Specifically, choose any two anticommuting (i.e. perpendicular) imaginary units $\hat{r}$ and $\hat{s}$ which lie in the plane of the desired rotation. Then if the desired amount of rotation in that plane is 2θ, do two flips around the two directions $\hat{r}$ and $\cos\theta\ \hat{r} + \sin\theta\ \hat{s}$ (which are separated by the angle θ). To do this, we define the composition $\phi^{(2)}_{(\hat{r},\hat{s},\theta|\alpha)}$ via

$$\phi^{(2)}_{(\hat{r},\hat{s},\theta|\alpha)} := \phi_{(\cos\theta\ \hat{r}+\sin\theta\ \hat{s},\alpha)} \circ \phi_{(\hat{r},-\alpha)} \tag{3.25}$$

in particular, for $\alpha = \frac{\pi}{2}$:

$$\begin{aligned}&\phi^{(2)}_{(\hat{r},\hat{s},\theta|\frac{\pi}{2})}(x) := \\ &\exp\left(\tfrac{\pi}{2}(\cos\theta\ \hat{r} + \sin\theta\ \hat{s})\right) \left[\exp\left(-\tfrac{\pi}{2}\,\hat{r}\right) x \exp\left(\tfrac{\pi}{2}\,\hat{r}\right)\right] \exp\left(-\tfrac{\pi}{2}(\cos\theta\ \hat{r} + \sin\theta\ \hat{s})\right)\end{aligned} \tag{3.26}$$

where the superscript "(2)" indicates the number of simple axis-angle ϕ's involved in the composition. In order to understand why $\phi^{(2)}$ works, consider its effects on different subspaces. In the plane spanned by $\hat{r}$ and $\hat{s}$, $\phi^{(2)}$ is just the composition of two reflections with respect to the two directions $\hat{r}$ and $\cos\theta\ \hat{r}+\sin\theta\ \hat{s}$ as mirror lines, amounting to a total rotation by 2θ, so that θ is indeed the continuously changing parameter. In particular $\phi^{(2)}_{(\hat{r},\hat{s},0)} = \mathbf{1}$. In the direction orthogonal to the plane, the flips are in opposite directions and therefore cancel. We call $\phi^{(2)}$ the plane-angle form of the rotations because it parameterizes rotations in terms of their plane and angle. In the case of the quaternions we can of course use the group homomorphism property of the ϕ's to express $\phi^{(2)}$ as a single ϕ:

$$\phi^{(2)}_{(\hat{r},\hat{s},\theta|\frac{\pi}{2})} = \phi_{(\cos\theta\ \hat{r}+\sin\theta\ \hat{s},\frac{\pi}{2})} \circ \phi_{(\hat{r},-\frac{\pi}{2})} = \phi_{(\hat{r}\hat{s},\theta)} \tag{3.27}$$

since

$$\exp\left(\frac{\pi}{2}(\cos\theta\ \hat{r}+\sin\theta\ \hat{s})\right)\exp\left(-\frac{\pi}{2}\hat{r}\right)=(\cos\theta\ \hat{r}+\sin\theta\ \hat{s})(-\hat{r})=\cos\theta+\sin\theta\ \hat{r}\hat{s} \tag{3.28}$$

We see that $\phi^{(2)}_{(\hat{r},\hat{s},\theta|\frac{\pi}{2})}$ only depends on the product $\hat{r}\hat{s}$, which in turn depends only on the plane (and orientation) of $\hat{r}$ and $\hat{s}$. Therefore any pair of anticommuting units spanning the same plane with the same orientation may replace $\hat{r}$ and $\hat{s}$ without changing the combined transformation.

We have seen that Φ maps all of S^3 to $Aut(\mathbb{H})$, but this new parameterization of the rotations only uses q's of the form $\exp\left(\frac{\pi}{2}\hat{r}\right)$, i.e. the angle in each of the individual flips is always the constant $\frac{\pi}{2}$.[10] This means that just a single S^2 slice of S^3 (the equator) maps under Φ to a generating set for $Aut(\mathbb{H})$.

3.3.2. Octonions and $SO(7)$

Now let us examine the more complicated case, $\mathbb{K}_8=\mathbb{O}$. We notice that for the octonions each line in the triangle, and more generally each associative triple of anticommuting, purely imaginary octonions of modulus 1, is just a copy of the imaginary units $\{e_2,e_3,e_4\}$ in $\mathbb{H}$. Therefore, if we consider the same conjugation map as we did in the quaternionic case with $q=\exp(\theta\, e_2)$, we obtain the associated matrix $A_{(e_2,\theta)}$:

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[10]Because $(-\theta)\hat{r}$ can be interpreted as $\theta(-\hat{r})$, the choice of the sign of the angle in each flip has no consequences. Therefore we have chosen the signs in (3.26) (and in later sections) for convenience.

$$A_{(e_2,\theta)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos 2\theta & -\sin 2\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & \sin 2\theta & \cos 2\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos 2\theta & -\sin 2\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin 2\theta & \cos 2\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cos 2\theta & -\sin 2\theta \\ 0 & 0 & 0 & 0 & 0 & 0 & \sin 2\theta & \cos 2\theta \end{bmatrix} \tag{3.29}$$

We see that this transformation yields three simultaneous rotations by an angle of 2θ in three mutually orthogonal planes which are all orthogonal to e_2. The pairs of imaginary units which are rotated into each other are just the pairs which each form an associative triple with e_2. Moreover, since the rotations in the three planes are equal, the choice of these planes is not unique.

For an arbitrary $\hat{r}$ we can always find a (nonunique) set of 3 pairwise orthogonal planes, orthogonal to $\hat{r}$, such that $\phi_{(\hat{r},\theta)}$ represents an axis-angle rotation in each of the quaternionic subspaces spanned by one of the planes and $\hat{r}$. For the special case $\theta = \frac{\pi}{2}$, $A_{(\hat{r},\theta)}$ has 8 real eigenvalues, 6 of which are -1. In this case the extra degeneracy means that if we choose $\hat{r}$ anywhere on, for example, the 2-3-4 subspace the effect on the 5-6 and 7-8 planes is the same.

Because each ϕ_q rotates three planes, it looks naively as if we should only be able to describe a subset of $SO(7)$ in this way. Surprisingly, this is not true. We can in fact describe all of $SO(7)$ and it turns out that the non-associativity of multiplication in $\mathbb{O}$ plays a crucial role. For $\mathbb{K}_8 = \mathbb{O}$, $\phi_p \circ \phi_q \neq \phi_{pq}$ in general, i.e. Φ is not a group homomorphism. In fact, $\phi_p \circ \phi_q \neq \phi_r$, for **any** $r \in \mathbb{O}$ unless $\operatorname{Im} p$ and $\operatorname{Im} q$ point in the same direction. It is this fact which allows $\Phi(S^7)$ to generate a Lie group with dimension larger than 7. For instance, by using more than one mapping, we can give explicit expressions for all of the elementary rotations. An elementary rotation in the i-j plane, for example, is given by $\phi_p \circ \phi_{q^*} \circ \phi_p \circ \phi_q$, where $q = \exp(\theta\, e_k)$, $p = \exp\left(\frac{\pi}{2} e_i\right)$, $e_k = e_i e_j$. This yields a rotation by 4θ in the i-j plane. The extra transformations undo the rotation in the other two planes, which were initially rotated by ϕ_q. The elementary rotations generate all of $SO(7)$.

Alternatively, the plane-angle form of the quaternionic case (involving only rotations with $\theta = \frac{\pi}{2}$) goes through as before, since in all the directions orthogonal to both axes the two rotations by π still cancel. Therefore $\phi^{(2)}_{(e_i, e_j, \theta | \frac{\pi}{2})}$ is another way of expressing a rotation by 2θ in the i-j plane. We see from the axis-angle form of the rotations that Φ maps the unit sphere in $\mathbb{O}$ to a generating set of $SO(7)$. As the plane-angle form shows, the equatorial S^6 is actually sufficient to provide a generating set of $SO(7)$.

3.3.3. Octonions and G_2

In the octonionic case we have obtained a larger group than we were looking for; all of $SO(7)$ instead of only its subgroup (of automorphisms of the octonions) G_2. However, we shouldn't have expected ϕ_q to be an automorphism since (3.16) is equivalent to

$$(qxq^{-1})(qyq^{-1}) = q(xy)q^{-1} \tag{3.30}$$

which would require the q's in between x and y to cancel. (3.30) only holds in general if multiplication is associative; but for certain choices for q, ϕ_q might still be an automorphism. For $q = \exp(\theta\, e_2)$, we find that (3.30) places no restriction on θ if e_2, $\operatorname{Im} x$, and $\operatorname{Im} y$ lie on one line in the triangle (when the calculation reduces to the quaternionic case). However, if e_2, x, and y contain anti-associative components, their products are not equal on the two sides of (3.30). Instead we obtain the following two equations for θ:

$$\begin{aligned} \cos 4\theta &= \cos 2\theta \\ -\sin 4\theta &= \sin 2\theta \end{aligned} \tag{3.31}$$

The solutions for $\angle$ are $\theta = k\frac{\pi}{3}$, $k = 0, \ldots, 5$. Obviously, e_2 can be replaced by any purely imaginary octonionic unit. Hence a single mapping, ϕ_q, is an automorphism of $\mathbb{O}$ if and only if

$$q = \exp\left(k\frac{\pi}{3}\hat{r}\right), \qquad k = 0, \ldots, 5 \tag{3.32}$$

i.e. if and only if q is a sixth root of unity, $q^6 = 1$.

These maps are not all of the automorphisms of $\mathbb{O}$, but they do generate the whole group. As in the previous section, we need to consider compositions of ϕ_q's, this time satisfying (3.32). We will show that we can obtain all of G_2 in this way by checking that the dimension of the associated Lie algebra is correct. Notice that the set of allowed q's splits into four pieces depending on the value of $\mathrm{Re}\, q$, $\{\mathrm{Re}\, q = \pm 1, \pm\frac{1}{2}\}$. If $q = \pm 1$, then ϕ_q is the identity. The piece with $\mathrm{Re}\, q = -\frac{1}{2}$ is made up of points which are antipodal in S^7 to the piece with $\mathrm{Re}\, q = \frac{1}{2}$ (see Footnote 9). Therefore these two pieces contain the same maps and we only need to consider the piece with $\mathrm{Re}\, q = \frac{1}{2}$.

To determine the group that is generated by these maps, we consider compositions of maps of the form $\phi^{(2)}_{(i,j,\theta|\frac{\pi}{3})}$. These are flips involving angles of $\frac{\pi}{3}$ so that each individual ϕ is an automorphism (instead of $\frac{\pi}{2}$ as in the last section). Of course, $\phi^{(2)}_{(i,j,0|\frac{\pi}{3})} = \mathbf{1}$. Since $(\phi_q)^{-1} = \phi_{q^{-1}}$, we also see that the set of maps with $\mathrm{Re}\, q = \frac{1}{2}$ contains the inverse of each element. A dimensional analysis of the associated Lie algebra finds the dimension of the space spanned by

$$\left\{ \frac{\mathrm{d}}{\mathrm{d}\theta} \phi^{(2)}_{(i,j,\theta|\frac{\pi}{3})}\Big|_{\theta=0} : i,j = 2, \ldots, 8, i \neq j \right\} \tag{3.33}$$

to be 14 as follows. There are $7 \times 6 = 42$ choices for i and j. It turns out that the 6 choices belonging to one associative triple of units only give 3 linearly independent generators, which leaves us with 21. In addition three triples which have one unit in common also share one generator, which cuts the number down by 7 leaving us with 14 independent generators for the Lie algebra.[11] Therefore the group generated is a 14-dimensional subgroup of G_2, i.e. G_2 itself.

From the form of $\phi^{(2)}_{(i,j,\theta|\frac{\pi}{3})}$ we see that $\left\{\phi_q : q = \exp\left(\frac{\pi}{3}\hat{r}\right) = \frac{1}{2} + \frac{\sqrt{3}}{2}\hat{r}\right\} \cong S^6$ actually suffices as generating set for G_2. We saw in the previous subsection that Φ maps the equatorial S^6 to a generating set of $SO(7)$. Here we see that Φ maps a different S^6 slice of the octonionic unit sphere to a generating set of G_2.

3.3.4. Some Interesting Asides

As an interesting aside, we derive two new identities for commutators in $\mathbb{O}$ in the following way. Let $q = \frac{1}{2} + \frac{\sqrt{3}}{2}\hat{r}$ in (3.30). Then the terms containing $\sqrt{3}$ and not containing it must be equal independently. Thus we obtain

$$\begin{aligned} 4[\hat{r}, xy] &= (x - 3(\hat{r}x\hat{r}))[\hat{r}, y] + [\hat{r}, x](y - 3(\hat{r}y\hat{r})) \\ [\hat{r}, x][\hat{r}, y] &= xy - 4\hat{r}(xy)\hat{r} + x(\hat{r}y\hat{r}) + (\hat{r}x\hat{r})y - 3(\hat{r}x\hat{r})(\hat{r}y\hat{r}) \end{aligned} \tag{3.34}$$

where $x, y, \hat{r} \in \mathbb{O}$ with $\mathrm{Re}\,\hat{r} = 0$, $|\hat{r}| = 1$.

As another interesting aside, we note that if $q^6 = 1$ then $q^3 = \pm 1$ which implies ${\phi_q}^3 = \mathbf{1}$. This means that the set of elements of G_2 which are third roots of the identity generate G_2, because it contains all of the maps ϕ_q with $q^6 = 1$. But

[11]To do this analysis we returned to the matrix representation of G_2, (3.19), and used the computer algebra package MAPLE. The calculations are nontrivial, especially the proof that the remaining 14 generators are really independent. We were surprised by the result that the generator of $\phi^{(2)}_{(i,j,\theta|\frac{\pi}{3})}$ is not simply related to the generator of $\phi^{(2)}_{(j,i,\theta|\frac{\pi}{3})}$.

there are third roots of the identity map which are not given by any single ϕ_q with q in $\mathbb{O}^*/\mathbb{R}^*$. This is due to the fact that ϕ_q is determined completely by its fixed direction $\hat{r}$, whereas a third root of the identity map has more free parameters. For example, the following matrix is associated with an automorphism of $\mathbb{O}$ which fixes e_2 and its third power is the identity, but it is not equal to A_q with $q = \exp\left(\frac{\pi}{3}(\pm e_2)\right)$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cos\frac{2\pi}{3} & \sin\frac{2\pi}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{bmatrix} \tag{3.35}$$

A similar statement holds for the generating set of $SO(7)$ which we found. It contains maps which square to the identity, because we had $q = \exp\left(\frac{\pi}{2}\,\hat{r}\right)$ whence $q^2 = -1$. But again not all the elements of $SO(7)$ which square to the identity are given as a ϕ_q.

3.4. MORE ISOMETRIES

Due to (3.10), we see that multiplying an element of $\mathbb{H}$ or $\mathbb{O}$ by an element of modulus 1 is always an isometry. The isometries of the previous section ($SO(n-1)$ and $Aut(K_n)$ for $n = 4, 8$) were all obtained using the asymmetric product, $\phi_q(x) = qxq^{-1}$. In this section we examine two other classes of isometries on $\mathbb{H}$ and $\mathbb{O}$.

3.4.1. Symmetric Products

First we show that it is possible to describe all of $SO(n)$ for $n = 4, 8$ using symmetric products. We define

$$\Psi : \{q \in \mathbb{K}_n : |q| = 1\} \to L(\mathbb{K}_n, \mathbb{K}_n) \tag{3.36}$$
$$q \mapsto \psi_q = \psi_{(\hat{r},\theta)} : \mathbb{K}_n \to \mathbb{K}_n$$
$$x \mapsto qxq = \exp(\theta\,\hat{r})\,x\,\exp(\theta\,\hat{r})$$

As with the conjugation maps, this is well-defined even for $\mathbb{K}_8 = \mathbb{O}$, since the associator $[q, x, q]$ vanishes. As before $(\psi_q)^{-1} = \psi_{q^{-1}}$ and $(\psi_q)^2 = \psi_{q^2}$ hold. We also note that $\psi_q = \psi_{-q}$ and that ψ_q is linear.

This isometry, however, does not fix the reals. We denote the matrix associated with ψ_q by B_q, where $\psi_q(x) = (B_q)^i{}_j x^j e_i$. Then $B_q \in SO(n)$ since $\psi_q = (\psi_{\sqrt{q}})^2$ (equivalently, $B_q = (B_{\sqrt{q}})^2$). Letting $q = \exp(\theta\,e_2)$, we obtain

$$B_{(e_2,\theta)} = \begin{bmatrix} \cos 2\theta & -\sin 2\theta & 0 & \dots & 0 \\ \sin 2\theta & \cos 2\theta & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \tag{3.37}$$

This is just a rotation by 2θ in the 1-2 plane. Similarly, any rotation by 2θ in the plane spanned by e_1 and any imaginary unit $\hat{r}$ is given by ψ_q with $q = \exp(\theta\,\hat{r})$.

But what about rotations in the purely imaginary subspace, $SO(n-1)$? Recall from the last section that the plane-angle construction of the elementary rotations in $SO(n-1)$ used a composition of two flips $\phi_p \circ \phi_q$ where p and q were both purely imaginary. But notice that $\psi_q = -\phi_q$ when q is imaginary, i.e. when $\theta = \frac{\pi}{2}$. Thus the maps $\{\psi_q : \; q = \exp\left(\frac{\pi}{2}\,\hat{r}\right), \operatorname{Re}\hat{r} = 0, |\hat{r}| = 1\}$ generate a group which includes $SO(n-1)$. Since we already found the rotations involving the real part we see that $\Psi(S^{n-1})$ generates all of $SO(n)$.

It is worth noting that the ψ_q's work differently from the ϕ_q's. For a single ψ_q, q is in the plane of rotation, whereas for a single ϕ_q, q was a fixed direction. Also, $\psi_p \circ \psi_q(x) = p(qxq)p \neq \psi_{pq} = (pq)x(pq)$, even for $\mathbb{H}$, since the order of the products is different. Therefore Ψ is not a group homomorphism.

However the Moufang identities (3.3) do demonstrate a partial group homomorphism property by providing a way of combining three ψ's together into a single ψ in some cases. For arbitrary $p, q \in \mathbb{K}_n$, with $|q| = |p| = 1$,

$$\psi_q \circ \psi_p \circ \psi_q = \psi_{qpq} \qquad \text{since} \qquad q\,(p\,(qxq)\,p)\,q = (qpq)\,x\,(qpq) \;\; \forall x \in \mathbb{K}_n \tag{3.38}$$

For any anticommuting imaginary units $\hat{r}$ and $\hat{s}$, the following identity is straightforward to prove:

$$\exp(\theta\,\hat{s}) = \exp\left(-\frac{\pi}{4}\,\hat{r}\right)\exp\left(\frac{\pi}{2}(\cos\theta\,\hat{r} + \sin\theta\,\hat{s})\right)\exp\left(-\frac{\pi}{4}\,\hat{r}\right) \tag{3.39}$$

Together with (3.38), (3.39) shows that a rotation $\psi_{(e_i,\theta)}$ in the 1-i plane by an arbitrary angle 2θ can be described as a combination of flips of fixed angle:

$$\psi_{(e_i,\theta)} = \psi_{(\hat{r},-\frac{\pi}{4})} \circ \psi_{(\cos\theta\,\hat{r}+\sin\theta\,e_i,\frac{\pi}{2})} \circ \psi_{(\hat{r},-\frac{\pi}{4})} \tag{3.40}$$

where $\hat{r}$ is any imaginary unit which anticommutes with e_i. (3.40) uses flips of angle $\frac{\pi}{2}$ and $\frac{\pi}{4}$. But since a flip with an angle of $\frac{\pi}{2}$ can be written as the square of a flip with angle $\frac{\pi}{4}$ and since we were able to write $SO(n-1)$ in terms of flips with angle $\frac{\pi}{2}$, we can write all of $SO(n)$ in terms of flips of fixed angle $\frac{\pi}{4}$. Therefore the image under Ψ of an $S^{n-2} \cong \{q = \exp\left(\frac{\pi}{4}\,\hat{r}\right) : \operatorname{Re}\hat{r} = 0,\, |\hat{r}| = 1\}$ slice of S^n suffices to generate all of $SO(n)$.

To understand how (3.40) works, notice that the first flip rotates the real direction into some fairly arbitrary imaginary direction $\hat{r}$. The second flip then rotates this imaginary direction $\hat{r}$ with the physically significant imaginary direction $\hat{s}$. The last flip rotates the former real part back into place[12].

3.4.2. One-sided Multiplication

Now we consider one-sided multiplication. Of course, left multiplication and right multiplication with elements of modulus 1 together generate $SO(n)$ because, in particular, they generate the ψ_q's. But what about left multiplication alone? We define

$$\begin{aligned} \mathbf{X} : \{q \in \mathbb{K}_n : |q| = 1\} &\to L(\mathbb{K}_n, \mathbb{K}_n) \\ q \mapsto \chi_q = \chi_{(\hat{r},\theta)} &: \mathbb{K}_n \to \mathbb{K}_n \\ x &\mapsto qx = \exp(\theta\, \hat{r})\, x \end{aligned} \tag{3.41}$$

For both $\mathbb{H}$ and $\mathbb{O}$, we have $(\chi_q)^{-1} = \chi_{q^{-1}}$ and $(\chi_q)^2 = \chi_{q^2}$, since the associators $[q^{-1}, q, x]$ and $[q, q, x]$ vanish. The following relation, connecting the maps ϕ_q and ψ_q with χ_q, holds for the same reason:

$$\chi_q = \phi_{\sqrt{q}} \circ \psi_{\sqrt{q}} = \psi_{\sqrt{q}} \circ \phi_{\sqrt{q}} \tag{3.42}$$

Of course we can no longer identify antipodal points since $\chi_{-q} = -\chi_q \neq \chi_q$.

For the quaternions $\mathbf{X}$ is a group homomorphism, $\chi_p \circ \chi_q = \chi_{pq}$. So $\mathbf{X}(S^3)$ must be a 3-dimensional subgroup of $SO(4)$. Therefore, to investigate the structure of any χ_q on $\mathbb{H}$, it will be sufficient to consider χ_q with $q = \exp(\theta\, e_2)$. The associated matrix $C_{(e_2,\theta)}$ is

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[12]This sounds much like manipulations of the Rubik's Cube, which indeed inspired JS in part.

$$C_{(e_2,\theta)} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{bmatrix} \tag{3.43}$$

This transformation rotates two orthogonal planes by θ. For the general case $q = \exp(\theta\,\hat{r})$, the rotations are in the plane spanned by e_1 and $\hat{r}$ and the plane orthogonal to that, as can be seen from the relation (3.42) and our previous investigation of maps ϕ_q and ψ_q.

It is interesting that $\mathbf{X}(S^3)$ is **not** $SO(3)$, much less $SO(4)$. We might expect, then, that left multiplication for $\mathbb{K}_8 = \mathbb{O}$ would only describe a subgroup of $SO(8)$. Surprisingly this is not the case. It turns out that the non-associativity of octonionic multiplication allows left multiplication to generate all of $SO(8)$, as follows:

First we consider $\chi_{(e_2,\theta)}$. The associated matrix $C_{(e_2,\theta)}$ is:

$$C_{(e_2,\theta)} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 & 0 & 0 & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & \sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\theta & -\sin\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & 0 & 0 & 0 & 0 & \sin\theta & \cos\theta \end{bmatrix} \tag{3.44}$$

$\chi_{(\hat{r},\theta)}$ always rotates four planes by an angle θ. (This is also clear from (3.42) and the results of previous sections.)

Now suppose we want to do an elementary rotation in just one of these four planes. The key idea is that the composition of two maps (c.f. (3.26))

$$\chi^{(2)}_{(\hat{s},\hat{t},\theta|\frac{\pi}{2})}(x) := \exp\left(\frac{\pi}{2}(\cos\theta\,\hat{s} + \sin\theta\,\hat{t})\right)\left[\exp\left(-\frac{\pi}{2}\hat{s}\right)\, x\right] \tag{3.45}$$

where $\hat{s}\hat{t} = \hat{r}$, will rotate exactly the same four planes as the map $\chi_{(\hat{r},\theta)}$, but because of non-associativity the rotations will not all be in the same direction in both cases. In particular, the parts of x which anti-associate with s and t will be rotated in opposite directions in the two cases.

As an example, consider $C^{(2)}_{(3,4,\theta)}$, the matrix associated with $\chi^{(2)}_{(3,4,\theta|\frac{\pi}{2})}$:

$$C^{(2)}_{(3,4,\theta)} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 & 0 & 0 & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & \sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & 0 & 0 & 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix} \tag{3.46}$$

Within the associative portion $\{e_1, e_2 = e_3e_4, e_3, e_4\}$ the rotation indeed remains the same as in the previous example (3.44), but the orientation of the rotation in the other two planes is reversed.

Using these ideas, we find that an appropriate composition of $\chi_{(2,\theta)}$, $\chi^{(2)}_{(3,4,\theta)}$, $\chi^{(2)}_{(5,6,\theta)}$, and $\chi^{(2)}_{(7,8,\theta)}$ allows us to rotate any single plane of the four coordinate planes rotated by $\chi_{(e_2,\theta)}$. Notice that $e_3e_4 = e_5e_6 = e_7e_8 = e_2$, i.e. the combinations which appear are all the independent pairs which, in the multiplication triangle, multiply to the corner e_2. For example, $\chi_{(2,\theta)} \circ \chi^{(2)}_{(3,4,\theta)} \circ \chi^{(2)}_{(5,6,\theta)} \circ \chi^{(2)}_{(7,8,\theta)}$ rotates the 1-2 plane by an angle of 4θ. Similarly, $\chi_{(2,\theta)} \circ \chi^{(2)}_{(3,4,\theta)} \circ \chi^{(2)}_{(5,6,-\theta)} \circ \chi^{(2)}_{(7,8,-\theta)}$ rotates the 3-4 plane by the same amount.

In terms of the multiplication triangle we can give the following rules to determine the composition needed to do an elementary rotation in the i-j plane.

Suppose $i = 1$, then we need to choose the corner j for the single χ and the pairs on the lines leading to j for the three $\chi^{(2)}$'s. If neither i nor j is 1, the corner, i.e. the single χ part, is given by $e_k = e_i e_j$. The three $\chi^{(2)}$ pieces come from the pairs which multiply to e_k. The ij piece occurs in the standard orientation and the other two pairs reversed.

The infinitesimal versions of the two examples above show this structure even more clearly. For the first example, $x \mapsto x + \theta\,(e_2 x + e_3(e_4 x) + e_5(e_6 x) + e_7(e_8 x)) + \mathcal{O}(\theta^2)$; while for the second example, $x \mapsto x + \theta\,(e_2 x + e_3(e_4 x) - e_5(e_6 x) - e_7(e_8 x)) + \mathcal{O}(\theta^2)$. The infinitesimal version also provides a convenient way to count the dimension of the group. There are 7 units and 21 pairs of units yielding 28 independent generators of $SO(8)$. As advertised, we have produced all of $SO(8)$.

As with symmetric multiplication, the Moufang identities (3.3) imply that for any $q, p \in \mathbb{K}_n$, with $|q| = |p| = 1$,

$$\chi_q \circ \chi_p \circ \chi_q = \chi_{qpq} \tag{3.47}$$

Therefore we can write any $\chi_{(\hat{r},\theta)}$ as a series of flips with constant angle $\frac{\pi}{4}$ using (3.39) and (3.47):

$$\begin{aligned} \chi_{(\hat{r},\theta)}(x) &:= \exp(\theta\,\hat{r})\;x \\ &= \exp\left(-\tfrac{\pi}{4}\,\hat{s}\right)\left[\exp\left(\tfrac{\pi}{2}(\cos\theta\;\hat{s} + \sin\theta\;\hat{r})\right)\left[\exp\left(-\tfrac{\pi}{4}\,\hat{s}\right)\;x\right]\right] \end{aligned} \tag{3.48}$$

where $\hat{s}$ is any imaginary unit which anticommutes with $\hat{r}$.

From the second form of χ we see that $\mathbf{X}$, completely analogously to Ψ for $\mathbb{K}_8 = \mathbb{O}$, maps the same S^6 ($\cong \{q \in \mathbb{O} : q = \exp\left(\frac{\pi}{4}\,\hat{r}\right), \operatorname{Re}\hat{r} = 0, |\hat{r}| = 1\}$), now to a different generating set of $SO(8)$.

Right multiplication is completely analogous to left multiplication. The details can easily be worked out using $xq = (q^*\,x^*)^*$.

3.5. LORENTZ TRANSFORMATIONS

In $(3,1)$ spacetime dimensions, it is standard to use the isomorphism between $SO(3,1)$ and $SL(2,\mathbb{C})$ to write a vector as a 2×2 hermitian complex-valued matrix via

$$X^\mu \to X = \begin{pmatrix} x^+ & x \\ x^* & x^- \end{pmatrix} \tag{3.49}$$

where $x^\pm = x^0 \pm x^{n+1} \in \mathbb{R}$ are lightcone coordinates, $x = \sum_{i=1}^n x^i e_i \in \mathbb{K}_n$, and $n = 2$. The Lorentzian norm of X^μ is then given by[13]

$$X^\mu X_\mu = -\mathrm{det} X \tag{3.50}$$

Standard results on determinants of matrices with complex coefficients show that if X' is obtained from X by the unitary transformation

$$X' = MXM^\dagger \tag{3.51}$$

then

$$\begin{aligned} \det X' = \det(MXM^\dagger) &= \det M \det X \det M^\dagger \\ &= \det M \det M^\dagger \det X \\ &= |\det M|^2 \det X \\ &= \det(MM^\dagger)\,\det X \end{aligned} \tag{3.52}$$

Therefore, if the determinant of M has norm equal to 1, then $\det X' = \det X$ and (3.51) is a Lorentz transformation. Notice, however, that there is some redundancy. M can be multiplied by an arbitrary overall phase factor without altering the Lorentz

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[13]We use signature $(-1, +1, \ldots, +1)$

transformation since the phase in $M^\dagger$ will cancel the phase in M. To remove this redundancy, M is usually chosen to have determinant equal to 1 rather than norm 1, but this restriction is not necessary. In Appendix B we record explicit versions of M which give the elementary boosts and rotations. Any Lorentz transformation can be obtained from this generating set by doing more than one such transformation and since

$$X' = (M_n(...(M_1 X M_1^\dagger)...)M_n^\dagger) = (M_n...M_1)X(M_1^\dagger...M_n^\dagger) \tag{3.53}$$

we see that any finite Lorentz transformation can be implemented by a single transformation of type (3.51).

We can use (3.49), just as in the complex case, to write a vector in $(n+1,1)$ spacetime dimensions for $n = 4,8$ as a 2×2 hermitian matrix with entries in K_n. The extra quaternionic or octonionic components on the off diagonal correspond to the extra transverse spatial coordinates. The manipulations in (3.52) are no longer valid in these cases due to the non-commutativity and non-associativity of the higher dimensional division algebras, but the last expression on the right hand side is nevertheless equal to the left hand side. (Notice that it is also the only expression on the right hand side which is well-defined.) A quaternion or octonion valued matrix M which generates a finite Lorentz transformation in $(n+1,1)$ dimensions must satisfy $\det(MM^\dagger) = 1$. An octonion valued matrix M must also satisfy an additional restriction which ensures that the transformation on the right hand side of (3.51) is well-defined[14].

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[14]The conditon that X' be hermitian is identical to the condition that there be no associativity ambiguity in (3.51). Both of these things will be true if and only if $\operatorname{Im} M$ contains only one octonionic direction or if the columns of $\operatorname{Im} M$ are real multiples of each other.

Looking at the elementary boosts and rotations in Appendix B, we see that for the quaternionic or octonionic cases if we simply let $e_2 \longrightarrow e_i$, for $i = 2, \ldots, n$, then we get all of the new boosts and some of the new rotations. The rotations which are missing are just the ones which rotate the purely imaginary parts of x into each other. But now consider a transformation with $M = q\mathbf{1} = \exp(\theta\,\hat{r})\mathbf{1}$, where $|q| = 1$. Since the diagonal elements $x^{\pm}$ of X are real, they are unaffected by these phase transformations. The off-diagonal elements, however, transform by a conjugation map:

$$x \longmapsto qxq^* \tag{3.54}$$

As we saw in Section 3, these conjugation maps give all of $SO(3)$ in the quaternionic case, and if repeated maps are included they give all of $SO(7)$ in the octonionic case. This is just what we needed. In the $(3,1)$ dimensional complex case the phase freedom is just the residue left over from these extra rotations which occur when there is more than one imaginary direction.

So we have shown that **all** finite Lorentz transformations can be implemented explicitly as in (3.51), simply by doing several such transformations in a row:

$$X' = (M_n(...(M_1 X M_1^\dagger)...)M_n^\dagger) \tag{3.55}$$

Since the octonions are not associative, (3.55) is **not** the same as

$$X' = (M_n...M_1)X(M_1^\dagger...M_n^\dagger) \tag{3.56}$$

and it is precisely this non-associativity which means that there is enough freedom in (3.55) to obtain **any** finite Lorentz transformation.

3.6. DISCUSSION

First we described $SO(3)$ using quaternions and $SO(7)$ using octonions via (a series of) conjugation maps, namely the maps ϕ_q with $q = \exp(\theta\,\hat{r})$. We obtained $Aut(O)$ ($\cong G_2$) by restricting θ to be $\frac{\pi}{3}$. Then we described $SO(4)$ using quaternions and $SO(8)$ using octonions via the symmetric maps ψ_q and also $SO(8)$ using octonions via left multiplication χ_q. We suspect that the existence of two different descriptions of $SO(8)$ is related to triality of the octonions.

It is worth reiterating here that our implementation of the symmetry groups of $\mathbb{H}$ and $\mathbb{O}$ provides an interesting new twist on the interpretation of rotations. The usual way of looking at a finite rotation is that a fixed axis is chosen and then the angle of rotation is changed continuously from zero until the desired rotation is achieved. Instead, the parameterizations in terms of flips presented in this paper use building blocks made of rotations with one fixed angle ($\frac{\pi}{2}$ for $SO(n-1)$ and $\frac{\pi}{4}$ for $SO(n)$). A finite rotation is accomplished by composing several such rotations, all with the same fixed angle. The relationship of the various axes in the composition is varied from initial alignment until the desired rotation is achieved. We used these flips to exhibit generating sets for $SO(8)$, $SO(7)$, and G_2 where each generating set is homeomorphic to a different S^6 subset of the octonionic unit sphere S^7. We believe that the parameterizations in terms of flips are new. In keeping with this point of view, the automorphisms of the octonions require flips with constant angle which is a multiple of $\frac{\pi}{3}$.

We then used the results for $SO(3)$ and $SO(7)$ to obtain an explicit description of finite Lorentz transformations on vectors in $(5,1)$ and $(9,1)$ dimensions in terms of unitary transformations on the 2×2 quaternionic or octonionic matrix representing the vectors. We believe that the finite version of $SL(2,\mathbb{O})$ requiring a succession of such unitary transformations is also new.

A number of other authors have attempted to find similar representations for the groups we have considered here. Conway [6] has independently developed the finite transformation rules for $SO(8)$ and $SO(7)$ (without flips), and for G_2. Ramond [7], gives a simple algebraic representation for the finite elements of G_2, $SO(7)$, and $SO(8)$, but uses a mixture of the various types of multiplication which we have used separately. A messy representation for the finite elements of G_2 and the infinitesimal elements of $SO(7)$ is given by Günaydin and Gürsey [8]. Finite transformations were used by Cartan and Schouten [9] to investigate absolute parallelisms on S^7. Coxeter [10] gives a special form for reflections with respect to a hyperplane in $\mathbb{R}^8$. Infinitesimal transformations are found more frequently [11]. A detailed analysis can be found in [12] where generators of $SO(8)$, $SO(7)$, and G_2 are given in terms of octonions. Their relation to integrated transformations is indicated but the actual integration is not carried out.

3.7. ACKNOWLEDGMENTS

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4. THE GENERAL CLASSICAL SOLUTION OF THE SUPERPARTICLE

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The theory of vectors and spinors in 9+1 dimensional spacetime is introduced in a completely octonionic formalism based on an octonionic representation of the Clifford algebra $Cl(9,1)$. Part of the Fierz matrix is derived in this framework. Then the general solution of the classical equations of motion of the CBS superparticle is given to all orders of the Grassmann hierarchy. Finally a spinor and a vector are combined into a 3×3 Grassmann, octonionic, Jordan matrix in order to construct a superspace variable to describe the superparticle. The combined Lorentz and supersymmetry transformations of the fermionic and bosonic variables are expressed in terms of Jordan products.

4.1. INTRODUCTION

The relationship between the division algebras and the existence of supersymmetric theories has been observed before especially in the context of string theory [1,2]. In particular, the division algebras have been used to solve the classical equations of motion in these models [3]. However, there have been difficulties in the octonionic case because of the non-associativity of this division algebra of highest dimension. For example, the Lorentz invariance of the formalism was unclear. Since the CBS superparticle [4] is an ideal testing ground to introduce techniques using division algebras and explore supersymmetry, it has attracted some attention. I. Oda *et al.* [5] and H. Tachibana & K. Imaeda [6] have answered some of the questions in this area. The connection of supersymmetric theories to the division algebras can also be made in terms of Jordan algebras as in [7] for the superstring and the superparticle. But again the implementation in the concrete case of the CBS superparticle or the superstring [8] did not yield a superior formalism, that, for example, made the symmetries of the model transparent.

This article carries on the previous attempts to cast the theory in a form that clearly displays its symmetries. However, a more transparent and powerful octonionic formalism is used. We show that one of the important Fierz [9] identities for the superstring reduces to the alternative property of the octonions. In addition, we go beyond a mere rewriting of vector and spinor variables in terms of octonionic expressions, with supersymmetry and Lorentz transformations acting differently on these variables. We succeed in introducing a unified superspace variable as a Jordan matrix, that includes both fermionic and bosonic variables. Both the supersymmetry transformations and the general solution are expressed in terms of Jordan matrices involving both kinds of variables in this unified way. We are aware of other work [10] related to the topics in this article.

The article is organized as follows. Section 4.2 introduces the octonionic formalism for vectors and spinors and their Lorentz transformations in 9+1 dimensions. (This article deals exclusively with the 9+1-dimensional case. The analogues in 5+1, 3+1, and 2+1 dimensions can be found easily.) A subsection using the octonionic analogue of the Fierz-matrix derives what we call the 3-Ψ's rule, an identity that is needed for the Green-Schwarz superstring to be supersymmetric [11]. A note on the notion of octonionic dotted and undotted spinors concludes this introductory section. Section 4.3 derives the general classical solution of the equations of motion for the CBS superparticle. Section 4.4 develops the Jordan matrix formalism combining bosonic and fermionic variables into one object. Lorentz and supersymmetry transformations and the superparticle action are expressed in this way.

4.2. OCTONIONIC SPINORS AND THE 3-Ψ'S RULE

4.2.1. Octonionic spinors

Octonionic spinors are based on an octonionic representation of a Clifford algebra. It is not obvious how to remove the obstacles arising from the non-associativity of the octonions. A rigorous treatment and resolution of this question can be found in [12], which also contains an introduction to octonions. Only general properties of octonions independent of a specific multiplication table will be used. However, because of the frequent use of octonionic identities, the reader may find more information on octonions as in [13,14] helpful.

The full Clifford algebra $Cl(9,1)$ in 9+1 dimensions has a real faithful irreducible representation in terms of 32×32-matrices. (As a reference for the general topic of Clifford algebras see [15].) An octonionic Majorana-Weyl representation is given in terms of 4×4-matrices:

$$\gamma_\mu = \begin{pmatrix} \mathbf{0} & \Gamma_\mu \\ \tilde{\Gamma}_\mu & \mathbf{0} \end{pmatrix}, \tag{4.1}$$

where

$$\begin{aligned} \Gamma_0 = & \ -\tilde{\Gamma}_0 = \mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ \Gamma_j = & \ \tilde{\Gamma}_j = \begin{pmatrix} 0 & e_j \\ e_j{}^* & 0 \end{pmatrix} \qquad (1 \le j \le 8), \\ \Gamma_9 = & \ \tilde{\Gamma}_9 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ \gamma_{11} = & \ \gamma_0\gamma_1 \ldots \gamma_9 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}. \end{aligned} \tag{4.2}$$

(In our convention an octonion x has real components x^j $(1 \le j \le 8)$, i.e., $x = x^j e_j$, where e_j $(1 \le j \le 8)$ are the octonionic units and $e_j{}^*$ their octonionic conjugates. For further information on octonions and octonionic identities, which will be used frequently, see [12,13]. The signature of the metric is $-+\ldots+$.) This representation is understood to act on a column of four octonions, a spinor, by left multiplication. This notion is necessary in order to remove the ambiguity that arises from the fact that octonionic multiplication is not associative. However, the fundamental property $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}$ remains valid under this interpretation. (For a rigorous treatment see [12].)

A vector with components x^μ $(0 \le \mu \le 9)$ is embedded in the Clifford algebra via

$$\not{x} = x^\mu \gamma_\mu = \begin{pmatrix} 0 & \mathbf{X} \\ \tilde{\mathbf{X}} & 0 \end{pmatrix}, \tag{4.3}$$

where

$$\mathbf{X} = x^\mu \Gamma_\mu \quad \text{and} \quad \tilde{\mathbf{X}} = x^\mu \tilde{\Gamma}_\mu. \tag{4.4}$$

(Boldface capitals always denote the 2×2 hermitian matrix associated with the vector denoted by the same lowercase letter.) The inverse of this relationship is

$$x^\mu = \frac{1}{4}\mathrm{Re\,tr}\,(\not{x}\gamma^\mu) = \frac{1}{4}\mathrm{Re\,tr}\left(\mathbf{X}\tilde{\Gamma}^\mu + \tilde{\mathbf{X}}\Gamma^\mu\right) = \frac{1}{2}\mathrm{Re\,tr}\left(\mathbf{X}\tilde{\Gamma}^\mu\right) = \frac{1}{2}\mathrm{Re\,tr}\left(\tilde{\mathbf{X}}\Gamma^\mu\right), \tag{4.5}$$

where indices are raised with the metric tensor g. Also note that

$$\Gamma^\mu = \tilde{\Gamma}_\mu \tag{4.6}$$

and

$$\tilde{\mathbf{X}} = \mathbf{X} - (\mathrm{tr}\,(\mathbf{X}))\,\mathbf{1}, \tag{4.7}$$

which implies

$$\mathbf{X}\tilde{\mathbf{X}} = \mathbf{X}^2 - (\mathrm{tr}\,(\mathbf{X}))\,\mathbf{X} = \tilde{\mathbf{X}}\mathbf{X} = -\det(\mathbf{X})\,\mathbf{1}, \tag{4.8}$$

since the characteristic polynomial for a hermitian 2×2-matrix A is $p_A(z) = z^2 - \mathrm{tr}\,(A)z + \det(A)$. This combination appears in the matrix product

$$\not{x}\not{x} = x_\mu x^\mu \mathbf{1}_{4\times 4}, \tag{4.9}$$

so that we have

$$x_\mu x^\mu \mathbf{1} = \mathbf{X}\tilde{\mathbf{X}} = -\det(\mathbf{X})\mathbf{1} \tag{4.10}$$

or its polarized form,

$$\begin{aligned} 2x_\mu y^\mu \mathbf{1} = & \ \mathbf{X}\tilde{\mathbf{Y}} + \mathbf{Y}\tilde{\mathbf{X}} \\ = & \ \tilde{\mathbf{X}}\mathbf{Y} + \tilde{\mathbf{Y}}\mathbf{X}. \end{aligned} \tag{4.11}$$

Now our convention for the numbering of the components of an octonion allows us to simply write

$$\mathbf{X} = \begin{pmatrix} x^+ & x \\ x^* & x^- \end{pmatrix} \text{ and } \tilde{\mathbf{X}} = \begin{pmatrix} -x^- & x \\ x^* & -x^+ \end{pmatrix}, \text{ where } x^\pm = x^0 \pm x^9. \tag{4.12}$$

A full spinor Ψ is given by a column of four arbitrary octonions. It can be decomposed into its positive and negative chiral projections,

$$\Psi_\pm := P_\pm \Psi, \tag{4.13}$$

via the projection operators

$$P_\pm = \frac{1}{2}(1 \pm \gamma_{11}). \tag{4.14}$$

For the chiral projections either the top or the bottom two components vanish. Depending on the context we will often regard a chiral spinor just as the column of the two non-vanishing components. We may also define the adjoint spinor:

$$\overline{\Psi} := \Psi^\dagger A, \tag{4.15}$$

. A is the matrix that intertwines the given representation with the hermitian conjugate representation:

$$\gamma_\mu^\dagger A = A\gamma_\mu. \tag{4.16}$$

Then A is up to a constant given by

$$A = \gamma_0\gamma_{11} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}. \tag{4.17}$$

(† denotes matrix transposition composed with octonionic conjugation.) The construction of a vector y out of two spinors Φ and Ψ is done in the usual way:

$$\begin{aligned} y_\mu := & \ \mathrm{Re}\left[\overline{\Phi}\gamma_\mu\Psi\right] \\ = & \ \mathrm{Re}\left[(\Phi_-{}^\dagger \ \ \Phi_+{}^\dagger)\begin{pmatrix} \mathbf{0} & \Gamma_\mu \\ \tilde{\Gamma}_\mu & \mathbf{0}\end{pmatrix}\begin{pmatrix}\Psi_+ \\ \Psi_-\end{pmatrix}\right] \\ = & \ \mathrm{Re}\left[\Phi_+{}^\dagger\tilde{\Gamma}_\mu\Psi_+\right] + \mathrm{Re}\left[\Phi_-{}^\dagger\Gamma_\mu\Psi_-\right]. \end{aligned} \tag{4.18}$$

So far we have built everything out of real octonions, i.e., the components x^j of an octonion $x = x^j e_j$ were real numbers. However, in order to consider anticommuting spinors we need to introduce elements of a Grassmann algebra. This notion can be incorporated into the octonionic formalism by letting the octonionic components take values in a real Grassmann algebra of arbitrary, possibly infinite dimension. Then the components of the octonions that make up an anticommuting spinor are odd Grassmannian. For the previous relation (4.18) we obtain

$$\begin{aligned} y_\mu = & \ \mathrm{Re}\left[\overline{\Phi}\gamma_\mu\Psi\right] \\ = & \ -\mathrm{Re}\left[\overline{\Psi}\gamma_\mu\Phi\right] \\ = & \ -\mathrm{Re}\left[\Psi_+{}^\dagger\tilde{\Gamma}_\mu\Phi_+\right] - \mathrm{Re}\left[\Psi_-{}^\dagger\Gamma_\mu\Phi_-\right]. \end{aligned} \tag{4.19}$$

The cyclic properties of the trace and the vanishing of the real parts of graded commutators and associators imply the following identities:

$$\begin{aligned} y_\mu = & \ -\mathrm{Re}\,\mathrm{tr}\left(\Psi\overline{\Phi}\gamma_\mu\right) = \mathrm{Re}\,\mathrm{tr}\left(\Phi\overline{\Psi}\gamma_\mu\right) \\ = & \ -\mathrm{Re}\,\mathrm{tr}\left(\Psi_+\Phi_+{}^\dagger\tilde{\Gamma}_\mu\right) - \mathrm{Re}\,\mathrm{tr}\left(\Psi_-\Phi_-{}^\dagger\Gamma_\mu\right) \\ = & \ \mathrm{Re}\,\mathrm{tr}\left(\Phi_+\Psi_+{}^\dagger\tilde{\Gamma}_\mu\right) + \mathrm{Re}\,\mathrm{tr}\left(\Phi_-\Psi_-{}^\dagger\Gamma_\mu\right) \\ = & \ \tfrac{1}{2}\mathrm{Re}\,\mathrm{tr}\left((\Phi_+\Psi_+{}^\dagger - \Psi_+\Phi_+{}^\dagger)\tilde{\Gamma}_\mu\right) + \tfrac{1}{2}\mathrm{Re}\,\mathrm{tr}\left((\Phi_-\Psi_-{}^\dagger - \Psi_-\Phi_-{}^\dagger)\Gamma_\mu\right). \end{aligned} \tag{4.20}$$

The full power of the octonionic formalism becomes evident, when we write y in terms of its Clifford representation $\mathbf{Y}$ and $\tilde{\mathbf{Y}}$ without the use of the Dirac matrices, as follows:

$$\begin{aligned} \mathbf{Y} &= (\Phi_+{\Psi_+}^\dagger - \Psi_+{\Phi_+}^\dagger) + (\Phi_-\widetilde{{\Psi_-}^\dagger - \Psi_-}{\Phi_-}^\dagger), \\ \tilde{\mathbf{Y}} &= (\Phi_+\widetilde{{\Psi_+}^\dagger - \Psi_+}{\Phi_+}^\dagger) + (\Phi_-{\Psi_-}^\dagger - \Psi_-{\Phi_-}^\dagger). \end{aligned} \tag{4.21}$$

This form of writing $\mathbf{Y}$ and $\tilde{\mathbf{Y}}$ involves the hermitian matrix product of two component spinors for which we have the following identity:

$$\begin{aligned} (\Phi_\sigma\widetilde{{\Psi_\sigma}^\dagger - \Psi_\sigma}{\Phi_\sigma}^\dagger) &= (\Phi_\sigma{\Psi_\sigma}^\dagger - \Psi_\sigma{\Phi_\sigma}^\dagger) - \left(\mathrm{tr}\left(\Phi_\sigma{\Psi_\sigma}^\dagger - \Psi_\sigma{\Phi_\sigma}^\dagger\right)\right) \\ &= (\Phi_\sigma{\Psi_\sigma}^\dagger - \Psi_\sigma{\Phi_\sigma}^\dagger) + ({\Psi_\sigma}^\dagger\Phi_\sigma - {\Phi_\sigma}^\dagger\Psi_\sigma)\,\mathbf{1}, \end{aligned} \tag{4.22}$$

where $\sigma \in \{+,-\}$. This relationship allows us to rewrite (4.21):

$$\begin{aligned} \mathbf{Y} &= (\Phi_+{\Psi_+}^\dagger - \Psi_+{\Phi_+}^\dagger) + (\Phi_-{\Psi_-}^\dagger - \Psi_-{\Phi_-}^\dagger) + ({\Psi_-}^\dagger\Phi_- - {\Phi_-}^\dagger\Psi_-)\,\mathbf{1}, \\ \tilde{\mathbf{Y}} &= (\Phi_+{\Psi_+}^\dagger - \Psi_+{\Phi_+}^\dagger) + ({\Psi_+}^\dagger\Phi_+ - {\Phi_+}^\dagger\Psi_+)\,\mathbf{1} + (\Phi_-{\Psi_-}^\dagger - \Psi_-{\Phi_-}^\dagger). \end{aligned} \tag{4.23}$$

These identities are plausible because of equation (4.20). To prove them we need to use the fact that the Γ_μ are a basis for the space of the hermitian matrices. (Note that the matrices are grouped so that the combinations in parentheses are hermitian, in particular, their traces are real, which means that we may commute octonionic products and/or take octonionic conjugates.)

4.2.2. Lorentz transformations

In Clifford language the orthogonal group is the Clifford group which is generated by unit vectors. A unit vector p induces a reflection at a line both on vectors and on spinors via the transformations

$$\begin{aligned} \not{x} &\to \not{p}\not{x}\not{p}^{-1}, \\ \Psi &\to \not{p}\Psi. \end{aligned} \tag{4.24}$$

Vectors parallel to p remain fixed, whereas those perpendicular to p are inverted. A pair of unit vectors p, q induces a rotation in the plane spanned by them, which means that the even part of the Clifford group corresponds to the simple orthogonal group. Specifically the 9+1-dimensional proper orthochronous Lorentz transformations are generated by

$$\begin{aligned} \mathbf{X} &\rightarrow \mathbf{P}(\tilde{\mathbf{Q}}\mathbf{X}\tilde{\mathbf{Q}})\mathbf{P}, \\ \Psi_+ &\rightarrow \mathbf{P}(\tilde{\mathbf{Q}}\Psi_+), \\ \Psi_- &\rightarrow \tilde{\mathbf{P}}(\mathbf{Q}\Psi_-), \end{aligned} \tag{4.25}$$

where $p_\mu p^\mu = q_\mu q^\mu = 1$. More details specifically about the effects of the non-associativity of the octonions are given in [12,13].

4.2.3. The 3-Ψ's rule

The previous relationships (4.23), which represent part of the octonionic analogue of the Fierz identities, allow us to deduce the 3-Ψ's rule for anticommuting 9+1-D Majorana-Weyl spinors: (We take $\Psi_k = P_+\Psi_k$ for $k = 1, 2, 3$.)

$$\begin{aligned} \gamma^\mu\Psi_1\overline{\Psi_2}\gamma_\mu\Psi_3 &= (\widetilde{\Psi_2\Psi_3{}^\dagger - \Psi_3\Psi_2{}^\dagger})\Psi_1 \\ &= (\Psi_2\Psi_3{}^\dagger - \Psi_3\Psi_2{}^\dagger)\Psi_1 - \left(\mathrm{Re}\,\mathrm{tr}\left(\Psi_2\Psi_3{}^\dagger - \Psi_3\Psi_2{}^\dagger\right)\right)\Psi_1 \\ &= (\Psi_2\Psi_3{}^\dagger)\Psi_1 - (\Psi_3\Psi_2{}^\dagger)\Psi_1 + \Psi_1(\Psi_3{}^\dagger\Psi_2) - \Psi_1(\Psi_2{}^\dagger\Psi_3). \end{aligned} \tag{4.26}$$

When we add the terms generated by cyclic permutations of the spinors, we can express the result in terms of associators of octonions. We may even treat both spinor components simultaneously by defining an associator for spinors:

$$[\Psi_1, \Psi_2{}^\dagger, \Psi_3] := \Psi_1(\Psi_2{}^\dagger\Psi_3) - (\Psi_1\Psi_2{}^\dagger)\Psi_3 \tag{4.27}$$

This spinor associator is just a shorthand for the following expression involving associators of the components:

$$[\Psi_1, {\Psi_2}^\dagger, \Psi_3] = \begin{pmatrix} [\Psi_{11}, {\Psi_{21}}^*, \Psi_{31}] + [\Psi_{11}, {\Psi_{22}}^*, \Psi_{32}] \\ \\ [\Psi_{12}, {\Psi_{21}}^*, \Psi_{31}] + [\Psi_{12}, {\Psi_{22}}^*, \Psi_{32}] \end{pmatrix}, \tag{4.28}$$

where $\Psi_\alpha = \binom{\Psi_{\alpha 1}}{\Psi_{\alpha 2}}$ $(\alpha = 1, 2, 3)$. The associator for (non-Grassmann) octonions is an antisymmetric function of its three arguments. So the previous expression is symmetric in the last two anticommuting spinors Ψ_2 and Ψ_3, since their components are grouped together consistently:

$$[\Psi_1, {\Psi_2}^\dagger, \Psi_3] = [\Psi_1, {\Psi_3}^\dagger, \Psi_2]. \tag{4.29}$$

(The derivation is a little tricky and uses the fact that octonionic conjugation of one of the arguments of an associator merely changes its sign.) Therefore we see that

$$\begin{aligned} \gamma^\mu \Psi_1 \overline{\Psi_2} \gamma_\mu \Psi_3 + \text{ cyclic} &= -[\Psi_2, {\Psi_3}^\dagger, \Psi_1] + [\Psi_3, {\Psi_2}^\dagger, \Psi_1] + [\Psi_1, {\Psi_3}^\dagger, \Psi_2] \\ &\quad -[\Psi_1, {\Psi_2}^\dagger, \Psi_3] + [\Psi_2, {\Psi_1}^\dagger, \Psi_3] - [\Psi_3, {\Psi_1}^\dagger, \Psi_2] \qquad (4.30) \\ &= 0. \end{aligned}$$

This identity is required for the Green-Schwarz superstring to exhibit the global fermionic supersymmetry [11]. This derivation shows that the 3-Ψ's rule is a direct consequence of the alternativity of the octonionic algebra, i.e., the relevant part of the Fierz identities are naturally built into the algebraic structure of the octonions.

4.2.4. A note on dotted and undotted spinors

From the form (see (4.1) and (4.2)) of the octonionic representation of $Cl(9,1)$ one might suspect, that there is no essential difference compared to the analogous complex representation of $Cl(3,1)$, where we are familiar with the notion of dotted and undotted spinors with raised and lowered indices. This notion in the complex case arises from the fact, that in four dimensions complex conjugation of the Dirac matrices induces another faithful irreducible representation of the

Clifford algebra $Cl(3,1)$ and matrix transposition induces a faithful representation of the opposite Clifford algebra $Cl_{\rm opp}(3,1)$, i.e., the algebra obtained by defining $a_{\rm opp} \,{}_{\rm vopp}\, b_{opp} = (b \,{}_{\rm v}\, a)_{\rm opp}$, where ${}_{\rm v}$ (resp. ${}_{\rm vopp}$) denotes multiplication in the abstract algebra (resp. its opposite). Therefore, the two irreducible representations Γ and $\tilde{\Gamma}$ of the even subalgebra $Cl_0(3,1)$, are essentially just complex conjugates of each other, more precisely they are related by charge conjugation:

$$\tilde{\mathbf{X}}^{\dot{A}A} = \epsilon^{\dot{B}\dot{A}}(\mathbf{X}_{A\dot{A}})^* \epsilon^{AB}$$
$$\Longleftrightarrow \quad \tilde{\mathbf{X}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{X}^* \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{4.31}$$

This relationship still holds in the octonionic case, although octonionic conjugation does not result in another representation, nor does matrix transposition give a representation for the opposite Clifford algebra, because octonionic multiplication is not commutative:

$$(\not{p}\not{q})^* \neq \not{p}^*\not{q}^*, \tag{4.32}$$

$$(\not{p}\not{q})^T \neq \not{q}^T\not{p}^T. \tag{4.33}$$

As a consequence Φ, defined by

$$\Phi^{\dot{B}} := \epsilon^{\dot{B}\dot{A}}(\Psi_A)^* = \epsilon^{\dot{B}\dot{A}}\Psi_{\dot{A}}$$
$$\Longleftrightarrow \qquad \Phi := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Psi_+{}^*, \tag{4.34}$$

does not transform like a negative chirality spinor according to (4.25). For this reason we prefer to use the original relationship (4.7) as a definition. Remarkably, we will not ever have to use (4.31) in any derivation, which confirms that this definition is not of primary importance.

Hermitian conjugation is still an antiautomorphism:

$$(\not{p}\not{q})^\dagger = \not{q}^\dagger \not{p}^\dagger, \tag{4.35}$$

which we already used to obtain a Dirac hermitian form, which defines the spinor adjoint.

So only two pairs of the four spinor spaces with lowered/raised, undotted/dotted indices are in close correspondence, which allows only limited use of the (to some) familiar notation. This difference may be caused by the spinors being both Majorana and Weyl in 9+1 dimensions.

Actually, it is still possible to restore relations (4.32) and (4.33). Namely, one has to switch to the opposite octonionic algebra. For example, the octonionic conjugate of an octonionic representation is another representation, when the original octonionic product is replaced by its opposite. This idea of utilizing various multiplication rules of the octonions, for example, the rule for the opposite octonionic algebra, will be pursued further in [12].

4.3. THE SUPERPARTICLE ACTION, THE EQUATIONS OF MOTION AND THEIR SOLUTION

The action in the Lagrangian form or second order action for the CBS-superparticle [4] is given by

$$S = \int d\tau L(\tau), \tag{4.36}$$

where

$$\begin{aligned} L = &\ \tfrac{1}{2} e \pi_\mu \pi^\mu, \\ = &\ \tfrac{1}{4} e \operatorname{tr}\left(\Pi\tilde{\Pi}\right), \\ \pi_\mu = &\ e^{-1}[\dot{x}_\mu + \textstyle\sum_{A=1}^N \operatorname{Re} \overline{\dot{\theta}^A} \tilde{\Gamma}_\mu \theta^A] = e^{-1}[\dot{x}_\mu + \textstyle\sum_{A=1}^N \operatorname{Re} \overline{\dot{\theta}^A} \gamma_\mu \theta^A], \\ \Pi = &\ e^{-1}[\dot{\mathrm{X}} + \textstyle\sum_{A=1}^N (\dot{\theta}^A \theta^{A\dagger} - \theta^A \dot{\theta}^{A\dagger})], \end{aligned} \tag{4.37}$$

and the variables describing the superparticle are its spacetime position x_μ, a set of N Majorana-Weyl spinors θ^A, and e is the einbein or induced metric on the worldline. The following equations of motion are obtained from varying the action with respect to e

$$\begin{aligned} \pi_\mu \pi^\mu =& \; 0, \\ \iff \quad \operatorname{tr}\left(\mathbf{\Pi}\tilde{\mathbf{\Pi}}\right) =& \; 0; \end{aligned} \tag{4.38}$$

with respect to x

$$\begin{aligned} \dot{\pi}_\mu =& \; 0, \\ \iff \quad \dot{\mathbf{\Pi}} =& \; 0; \end{aligned} \tag{4.39}$$

and with respect to θ^A

$$\begin{aligned} \not{\pi}\dot{\theta}^A =& \; \pi_\mu \tilde{\Gamma}^\mu \dot{\theta}^A = 0, \\ \iff \quad \tilde{\mathbf{\Pi}}\dot{\theta}^A =& \; 0. \end{aligned} \tag{4.40}$$

We solve the algebraic equations for $\mathbf{\Pi}$ and $\dot{\theta}^A$. Equations (4.38) and (4.39) imply that π is a constant lightlike vector. Such vectors can be parametrized by 9 even Grassmann parameters $\{\pi_1, \ldots, \pi_9\}$ uniquely for the future or past light cone in the regular case, i.e., if at least one of these components is invertible and therefore has non-zero body. In this case $\sum_{i=1}^9 \pi_i^2$ is invertible and has up to a sign a unique square root π_0, whence π^+ or π^- is invertible.

Otherwise, in the singular case when all components of π have zero body, there may not exist any π_0 to make π_μ lightlike, or there may be multiple possibilities. (For example, if the spatial components are all zero, then π_0 may be any even Grassmann number which squares to zero. These difficulties arise, because $x \mapsto x^2$ is not injective in the neighborhood of zero.) We do not have a parametrization of this variation of the trivial solution.

In terms of even Grassmann octonions we parametrize $\mathbf{\Pi}$ by two real numbers $|a|, |b|$ and a unit octonion $\check{r}$, where $\check{r}\check{r}^* = 1$. This parametrization does not cover the cases where π^+ or π^- are not squares, which can happen if they are not invertible. If they are squares, but not both invertible, there may be other forms of π such that $\pi^+\pi^- - \pi\pi^* = 0$. If $|a| = 0$ or $|b| = 0$, then $\check{r}$ is undetermined:

$$\mathbf{\Pi} = \begin{pmatrix} |a|^2 & |a||b|\check{r} \\ |a||b|\check{r}^* & |b|^2 \end{pmatrix}. \tag{4.41}$$

Even in the pathological cases for $\mathbf{\Pi}$ we can solve (4.40) by letting

$$\dot{\theta}^A = \mathbf{\Pi}\zeta^A = \begin{pmatrix} \pi^+ \\ \pi^* \end{pmatrix} \zeta_1^A + \begin{pmatrix} \pi \\ \pi^- \end{pmatrix} \zeta_2^A, \tag{4.42}$$

where ζ^A is an odd Grassmannian spinor. This solution relies on the weak form of associativity, the so-called alternativity, of the octonions, which makes products which involve not more then two full octonions and their octonionic conjugates associative. If π^+ (resp. π^-) is invertible, we may redefine $\zeta_1^A \to \zeta_1^A - \frac{\pi}{\pi^+}\zeta_2^A$ (resp. $\zeta_2^A \to \zeta_2^A - \frac{\pi^*}{\pi^-}\zeta_2^A$) to see that our solution only depends on one arbitrary odd Grassmann octonion function.

If we can write $\mathbf{\Pi}$ as in (4.41), then

$$\dot{\theta}^A = \begin{pmatrix} |a| \\ |b|\check{r}^* \end{pmatrix} (|a|\zeta_1^A + |b|\check{r}\zeta_2^A) = \Psi_0\zeta_0^A, \tag{4.43}$$

where $\Psi_0 = \binom{|a|}{|b|\check{r}^*}$ is a commuting spinor and ζ_0 is an arbitrary odd Grassmann octonion function. So we gave the general classical solution for the CBS superparticle, except for a parametrization of the lightlike vector in the pathological cases.

In other solutions [5] $\mathbf{\Pi}$ is parametrized in terms of a commuting spinor $\Psi = \binom{a}{b}$:

$$\mathbf{\Pi} = \Psi\Psi^{\dagger} = \begin{pmatrix} |a|^2 & ab^* \\ ba^* & |b|^2 \end{pmatrix}. \tag{4.44}$$

This parametrization introduces a redundancy of 7 extra parameters that correspond to an octonionic unit sphere S^7, since only the combination $\breve{r} = \frac{ab^*}{|a||b|}$ enters into the off diagonal elements of $\mathbf{\Pi}$. Removing this redundancy reduces the number of octonionic directions to just the one of $\breve{r}$, which allowed us to find the general solution for $\dot{\theta}^A$ without much difficulty. So we started with one specific $\Psi_0 = \binom{|a|}{|b|\breve{r}^*}$ and obtained the general solution $\dot{\theta}^A = \Psi_0 \zeta_0^A$. The arbitrary Grassmann octonionic function ζ_0^A introduces only a second octonionic direction, so that all the products appearing in the expression $\tilde{\mathbf{\Pi}}\dot{\theta}^A$ are associative. For the general $\Psi = \binom{a}{b}$ we may want to try $\dot{\theta}^A = \Psi \xi^A$ with ξ^A an arbitrary odd Grassmann octonion, but we do not necessarily obtain a solution because of the non-associativity of the octonions.

A recent article by Cederwall & Preitschopf [16] proposes to modify the octonionic product to avoid the fixing of Ψ_0. Applying these ideas, we can give an alternate form of the solution:

$$\dot{\theta}^A = \Psi \circ_a \zeta_a^A \qquad \text{or} \qquad \dot{\theta}^A = \Psi \circ_b \zeta_b^A, \tag{4.45}$$

where $x \circ_a y := |a|^{-2}(xa^*)(ay)$ and similar for $\circ_b$. The new Grassmann functions ζ_a^A and ζ_b^A are related to ζ_0^A via $\frac{a}{|a|}\zeta_0^A = \zeta_a^A$ and $\frac{b}{|b|}\zeta_0^A = \zeta_b^A$. Again, there are difficulties if $|a|$ or $|b|$ are not invertible, because only one of the modified products exists in these cases.

In line with [16] the proper interpretation of $\mathbf{\Pi} = \Psi\Psi^{\dagger}$ is to view it as element of $\mathbb{R} \times \mathbb{O}P^1$, $\mathbb{O}P^1$ being the octonionic projective line. The sixteen parameters of Ψ are collapsed, using the Hopf [17] map: $\mathbb{R} \times S^{15} \approx \mathbb{R} \times S^8 \times S^7$. The singularities for $|a| = 0$ or $|b| = 0$ are caused by the fact that the particular coordinate maps cannot be extended to include both of these points. So the extra 7 parameters in

Ψ can be divided out, adapting the octonionic product. The modification of the product amounts to a rotation of the imaginary part modulo a automorphism of the octonionic product, which accounts for the octonionic unit sphere $S^7 \approx SO(7)/G_2$. In a sense the Lorentz invariance is already broken by specifying a certain multiplication rule of the octonionic product. The adaptation of the product to the spinor components can be seen to restore the Lorentz invariance.

From any of the forms for $\dot{\theta}^A$ we get θ^A by simply integrating the arbitrary odd Grassmann octonion function, using the form of (4.42), for example,

$$\theta^A = \Pi Z^A + \theta_0^A. \tag{4.46}$$

So θ^A is parametrized by an arbitrary Grassmann octonion function and a constant anticommuting spinor. $\mathbf{X}$ may now be computed by a simple integration. The local fermionic supersymmetry can be seen to provide a similar parametrization of the solutions as is shown in the next section.

4.4. THE JORDAN MATRIX FORMALISM

This section carries on the attempts of Foot & Joshi [8] and Gürsey [2]. We combine a fermionic spinor variable β and a bosonic vector $\mathbf{B}$ and scalar b into one superspace object, namely a 3×3 Jordan matrix $\mathcal{B}$:

$$\mathcal{B} = \begin{pmatrix} \mathbf{B} & \beta \\ \beta^\dagger & b \end{pmatrix}. \tag{4.47}$$

(β corresponds to a positive chirality spinor.) The Jordan product for Jordan matrices with Grassmannian entries is taken to be defined as in [8], which is equivalent to taking the hermitian part of the matrix product:

$$\mathcal{A} \circ \mathcal{B} := \frac{1}{2}\left(\mathcal{A}\mathcal{B} + (\mathcal{A}\mathcal{B})^\dagger\right). \tag{4.48}$$

The results of section 4.2.2 can be applied to obtain a generating set for all Lorentz transformations for a Jordan matrix:

$$\mathcal{A} \to \mathcal{M}\mathcal{A}\mathcal{M}^\dagger, \text{ where } \mathcal{M} = \begin{pmatrix} \mathbf{M} & 0 \\ 0 & 1 \end{pmatrix}, \ \mathbf{M} = \mathbf{P}\tilde{\Gamma}_1, \text{ and } p_\mu p^\mu = 1. \tag{4.49}$$

($\tilde{\mathbf{Q}}$ in (4.25) has been replaced by the constant $\tilde{\Gamma}_1$, which is purely real and allows us to move the parentheses. This subset of transformations, of course, still generates all of the Lorentz transformations.)

For the superparticle we consider as the fundamental superspace matrix

$$\mathcal{X} = \begin{pmatrix} \mathbf{X} & e^{\frac{1}{2}}\theta \\ e^{\frac{1}{2}}\theta^\dagger & e \end{pmatrix}. \tag{4.50}$$

(We already saw in the solution in the previous section that the fermionic variables decouple, which reflects a symmetry of the Lagrangian. In this section, we only consider one fermionic variable, i.e., $N = 1$.) The global supersymmetry transformation may then be written as

$$\begin{aligned} (1+\delta_\epsilon)\mathcal{X} &= \mathcal{Z}_\epsilon \circ \mathcal{X} \\ &= \begin{pmatrix} \mathbf{1} & 2e^{-\frac{1}{2}}\epsilon \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} \mathbf{X} & e^{\frac{1}{2}}\theta \\ e^{\frac{1}{2}}\theta^\dagger & e \end{pmatrix}. \\ &= \begin{pmatrix} \mathbf{X} + (\epsilon\theta^\dagger - \theta\epsilon^\dagger) & e^{\frac{1}{2}}(\theta+\epsilon) \\ e^{\frac{1}{2}}(\theta^\dagger + \epsilon^\dagger) & e \end{pmatrix}. \end{aligned} \tag{4.51}$$

Note that we used the non-hermitian matrix $\mathcal{Z}_\epsilon$ for this transformation, which avoids the extension to larger matrices as it is done in [2]. The λ-transformation has a simple structure as well:

$$
\begin{aligned}
(1+\delta_\lambda)\mathcal{X} = {} & \mathcal{X} \circ \mathcal{Z}_{\lambda\dot\theta}{}^\dagger \\
= {} & \begin{pmatrix} \mathbf{X} & e^{\frac{1}{2}}\theta \\ e^{\frac{1}{2}}\theta^\dagger & e \end{pmatrix} \circ \begin{pmatrix} \mathbf{1} & 0 \\ 2e^{-\frac{1}{2}}\lambda\dot\theta^\dagger & 1 \end{pmatrix} \\
= {} & \begin{pmatrix} \mathbf{X} + \lambda(\theta\dot\theta^\dagger - \dot\theta\theta^\dagger) & e^{\frac{1}{2}}(\theta + \lambda\dot\theta) \\ e^{\frac{1}{2}}(\theta^\dagger + \lambda\dot\theta^\dagger) & e \end{pmatrix}.
\end{aligned} \tag{4.52}
$$

We can also construct a superspace variable that contains the conjugate momentum $\mathbf{\Pi}$ of $\mathbf{X}$:

$$
\begin{aligned}
\mathcal{P}' := {} & e^{-1}\dot{\mathcal{X}} \circ \mathcal{Z}_\theta{}^\dagger \\
= {} & e^{-1}\begin{pmatrix} \dot{\mathbf{X}} & (e^{\dot{\frac{1}{2}}}\theta) \\ (e^{\dot{\frac{1}{2}}}\theta)^\dagger & \dot e \end{pmatrix} \circ \begin{pmatrix} \mathbf{1} & 0 \\ 2e^{-\frac{1}{2}}\theta^\dagger & 1 \end{pmatrix} \\
= {} & \begin{pmatrix} e^{-1}[\dot{\mathbf{X}} + (\dot\theta\theta^\dagger - \theta\dot\theta^\dagger)] & e^{-\frac{1}{2}}\dot\theta + \frac{3}{2}\dot e e^{-\frac{3}{2}}\theta \\ e^{-\frac{1}{2}}\dot\theta^\dagger + \frac{3}{2}\dot e e^{-\frac{3}{2}}\theta^\dagger & \dot e \end{pmatrix} \\
= {} & \begin{pmatrix} \mathbf{\Pi} & e^{-\frac{1}{2}}\dot\theta + \frac{3}{2}\dot e e^{-\frac{3}{2}}\theta \\ e^{-\frac{1}{2}}\dot\theta^\dagger + \frac{3}{2}\dot e e^{-\frac{3}{2}}\theta^\dagger & \dot e \end{pmatrix}.
\end{aligned} \tag{4.53}
$$

However, it seems to work better to postulate another superspace variable as the "conjugate" to $\mathcal{X}$:

$$
\mathcal{P} := \begin{pmatrix} \mathbf{\Pi} & e^{-\frac{1}{2}}\dot\theta \\ e^{-\frac{1}{2}}\dot\theta^\dagger & 0 \end{pmatrix} \tag{4.54}
$$

For it can be used to give a pretty form for the solution of the equations of motion:

$$
\mathcal{P} = (\phi_a\phi_a^\dagger)_{\circ_a} \quad \text{resp.} \quad \mathcal{P} = (\phi_b\phi_b^\dagger)_{\circ_b}, \tag{4.55}
$$

where

$$
\phi_a = \begin{pmatrix} a \\ b \\ e^{-\frac{1}{2}}\zeta_a{}^* \end{pmatrix} \quad \text{resp.} \quad \phi_b = \begin{pmatrix} a \\ b \\ e^{-\frac{1}{2}}\zeta_b{}^* \end{pmatrix}, \tag{4.56}
$$

and the products are evaluated using the modified product. Taking the hermitian part is implied, causing the (3,3) component to vanish. This form exactly reproduces (4.45). It can be interpreted to be a Grassmannian extension of the octonionic projective line, which can also be defined as the matrices which are idempotent up to scale:

$$\mathcal{P} \circ \mathcal{P} = (\mathrm{tr}\,(\mathcal{P}))\mathcal{P}. \tag{4.57}$$

(The Jordan product is understood to be based on the modified octonionic product.) A not quite so aesthetic form of the κ-transformation can also be obtained using $\mathcal{P}$:

$$\delta_\kappa \mathcal{X} = 4 \begin{pmatrix} \mathbf{0} & \theta \\ 0 & e^{\frac{1}{2}} \end{pmatrix} \circ \left(\mathcal{P} \circ \begin{pmatrix} \mathbf{0} & \kappa \\ \kappa^\dagger & 0 \end{pmatrix} \right). \tag{4.58}$$

Taking a closer look at this local fermionic symmetry, we realize that

$$\begin{aligned} \delta_\kappa \theta = \mathbf{\Pi}\kappa, \quad \delta_\kappa \mathbf{X} = \dot{\theta}\delta_\kappa\theta^\dagger - \delta_\kappa\theta\dot{\theta}^\dagger, \quad \delta_\kappa e = 2(\dot{\theta}^\dagger\kappa - \kappa^\dagger\dot{\theta}), \\ \Longrightarrow \quad \delta_\kappa \mathbf{\Pi} = 2[\kappa(\tilde{\mathbf{\Pi}}\dot{\theta})^\dagger - (\tilde{\mathbf{\Pi}}\dot{\theta})\kappa^\dagger], \end{aligned} \tag{4.59}$$

i.e., on shell $\delta_\kappa \mathbf{\Pi} = 0$ and $\delta_\kappa \theta$ has the form of the general solution for $\dot{\theta}$. (The form of the transformation simplifies due to our choice to include the scale in the definition of $\mathbf{\Pi}$.) So the κ-supersymmetry can be used to absorb the arbitrary odd Grassmann octonion function in the solution for θ, so that just a constant spinor remains. Therefore, acting with κ-transformations on the solutions of the form

$$\begin{aligned} \theta = {} & \theta_0, \\ \dot{\mathbf{X}} = {} & e\mathbf{\Pi} \\ \Longrightarrow \quad \mathbf{X} = {} & E\mathbf{\Pi} + \mathbf{X}_0, \end{aligned} \tag{4.60}$$

where θ_0 is a constant anticommuting spinor, $\mathbf{X}_0$ is a constant vector, $\mathbf{\Pi}$ is a constant lightlike vector, and $E = \int e(\tau)\, d\tau$ is the arclength along the worldline, generates all solutions.

The Freudenthal product for Jordan matrices is defined by

$$\mathcal{X}*\mathcal{Y} := \mathcal{X} \circ \mathcal{Y} - \frac{1}{2}\mathcal{X}(\mathrm{tr}\,(\mathcal{Y})) - \frac{1}{2}(\mathrm{tr}\,(\mathcal{X}))\mathcal{Y} + \frac{1}{2}\left[(\mathrm{tr}\,(\mathcal{X}))(\mathrm{tr}\,(\mathcal{Y})) - \mathrm{tr}\,(\mathcal{X} \circ \mathcal{Y})\right]\mathbf{1}. \tag{4.61}$$

This notion can be extended for Grassmannian Jordan matrices. The Lagrangian L for the superparticle is then given by the following form which has the same appearance as the E_6 invariant trilinear form on the non-Grassmannian Jordan algebra:

$$L = -\,\mathrm{tr}\,((\mathcal{P}*\mathcal{X}) \circ \mathcal{P}). \tag{4.62}$$

Due to the antisymmetry with respect to the spinor variables only the (3, 3) component of $\mathcal{X}$ contributes, i.e., $\mathcal{X}$ could be replaced by $\begin{pmatrix} \mathbf{0} & 0 \\ 0 & e \end{pmatrix}$ in (4.62):

$$L = -\,\mathrm{tr}\left((\mathcal{P}*\begin{pmatrix} \mathbf{0} & 0 \\ 0 & e \end{pmatrix}) \circ \mathcal{P}\right). \tag{4.63}$$

Also considering the components of $\mathcal{P}$ only the upper 2×2 matrix $\mathbf{\Pi}$ contributes, so that $\mathcal{P}$ can be replaced by $\mathcal{P}'$ in both forms of the Lagrangian. Alternate versions of this formalism, where the einbein e is substituted by 1 in the superspace variable $\mathcal{X}$, are possible maintaining the form of L, since the trilinear form only contains certain combinations of variables, i.e., products of two vectors and a scalar or of two spinors and a vector, both of which have units of length^2.

4.5. CONCLUSION

We have demonstrated the usefulness of the octonionic formalism in several ways in this article. We have solved the classical equations of motion for the CBS superparticle. The question of Lorentz covariance of the solution could be answered using a modified octonionic product. The local fermionic transformation could be seen to relate solutions and absorb the arbitrary odd Grassmann octonion function in the solution for the fermionic variable. We have been able to express Lorentz and all known supersymmetry transformations in terms of Jordan products involving Jordan matrices with Grassmannian entries. However, the exact form of the objects that should be used in these expressions was unclear because of the cancellations due to the anticommuting variables. We believe that an extension to the Green-Schwarz superstring will fix the form of the expressions, if such an extension is possible. Another interesting avenue is to explore the symmetries of the theory in terms of the Jordan matrices further. Taking a varying octonionic product into account, this may lead to similar generalizations of (super) Lie groups as the S^7 transformations in [16] are generalizations of group manifolds. An extension of the octonionic formalism off shell is needed to lead to a quantization of the theory in this formalism, but it may be the key to unlock the mysteries of the superstring.

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4.7. REFERENCES

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5. PERSPECTIVES

We have made respectable progress towards the formidable goal of establishing a foundation for octonionic descriptions of spacetime and supersymmetry. We have provided the basic framework for octonionic representations of Clifford algebras, we have given explicit examples for the relevant dimensions, and we started along a promising path toward a description of supersymmetry by understanding the triality maps and the supersymmetry transformations of the CBS superparticle.

More mathematically and physically interesting questions lie ahead. Are there octonionic descriptions of the other exceptional Lie groups similar to the one given for G_2? We have already found a piece of the description for the automorphism group of the exceptional Jordan algebra, F_4, namely the triality part. The way in which $SO(9,1)$ is embedded in E_6, acting on Jordan matrices, can be seen from the Lorentz transformations of the superspace variable of the superparticle. So this avenue looks promising.

Can we find a superspace variable for the Green-Schwarz superstring and express its supersymmetry transformations, as we did for the CBS superparticle? For the interesting case of $N = 2$ supersymmetry, the string contains two anticommuting Majorana-Weyl spinors. It is unclear whether both of these spinors should be combined into one object with the bosonic variable or whether two separate superspace variables should be introduced. Related to the two worldsheet dimensions of the string, we have two linearly independent partial derivatives. Because of this, cancellations involving the fermionic variable that occurred in the Lagrangian of the superparticle in the Jordan matrix form do not occur for the superstring.

Is there a natural extension of the octonionic formalism off shell? We have seen that octonions provide a natural parametrization of lightlike vectors or even of solutions of the superparticle, which correspond to the Hopf map $S^{15} \to S^8$. How can the extra dimension of an arbitrary, non-lightlike vector be introduced? The difficulty is to find the extension of the octonionic formalism off the lightcone that preserves its advantageous features. This step is necessary to utilize octonionic methods for a quantization of supersymmetric theories.

The answers to these questions may hold the key to our understanding of supersymmetric theories, and eventually the key to a theory of quantum gravity. We would be pleased to see this piece of work as a part of the mosaic that helps us understand the structure of this universe, space and time.

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APPENDICES

APPENDIX A

Structure matrices for our choice of multiplication rules for the octonions. (Note that if the sign of the first column is changed, the first matrix becomes $-\mathbf{1}$ and each matrix except the first becomes antisymmetric.)

$$[\Lambda^1{}_{jk}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

$$[\Lambda^2{}_{jk}] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix},$$

$$[\Lambda^3{}_{jk}] = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix},$$

$$[\Lambda^4{}_{jk}] = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix},$$

$$[\Lambda^5{}_{jk}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$[\Lambda^6{}_{jk}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$[\Lambda^7{}_{jk}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$[\Lambda^8{}_{jk}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

APPENDIX B

Using the following correspondence, which is explained in Section 3.5:

$$X^\mu \leftrightarrow X = \begin{pmatrix} x^+ & x \\ x^* & x^- \end{pmatrix}$$

we can write the elementary Lorentz transformations $L^\mu{}_\nu$ in terms of 2×2 hermitian matrices M over $\mathbb{K}_n$.

$$X'^\mu = L^\mu{}_\nu X^\nu \leftrightarrow X' = \begin{cases} MXM^\dagger, & \text{for } \textit{Categories 1} \text{ and } \textit{2} \\ M_2\left(M_1 X M_1^\dagger\right) M_2^\dagger, & \text{for } \textit{Category 3} \end{cases}$$

Category 1: Boosts

$X^0 \circlearrowleft X^1$:

$$L = \begin{pmatrix} \cosh\alpha & \sinh\alpha & 0 & \dots & 0 \\ \sinh\alpha & \cosh\alpha & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \leftrightarrow M = \begin{pmatrix} \cosh\left(\frac{\alpha}{2}\right) & \sinh\left(\frac{\alpha}{2}\right) \\ \sinh\left(\frac{\alpha}{2}\right) & \cosh\left(\frac{\alpha}{2}\right) \end{pmatrix}$$

$X^0 \circlearrowleft X^{n+1}$:

$$L = \begin{pmatrix} \cosh\alpha & 0 & \dots & 0 & \sinh\alpha \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \dots & 1 & 0 \\ \sinh\alpha & 0 & \dots & 0 & \cosh\alpha \end{pmatrix} \leftrightarrow M = \begin{pmatrix} \exp\left(\frac{\alpha}{2}\right) & 0 \\ 0 & \exp\left(-\frac{\alpha}{2}\right) \end{pmatrix}$$

$X^0 \circlearrowleft X^i$:

$$L = \begin{pmatrix} \cosh\alpha & 0 & \dots & 0 & \sinh\alpha & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 \\ \sinh\alpha & 0 & \dots & 0 & \cosh\alpha & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \end{pmatrix} \leftrightarrow \begin{matrix} M = \\ \begin{pmatrix} \cosh\left(\frac{\alpha}{2}\right) & e_i \sinh\left(\frac{\alpha}{2}\right) \\ -e_i \sinh\left(\frac{\alpha}{2}\right) & \cosh\left(\frac{\alpha}{2}\right) \end{pmatrix} \end{matrix}$$

Category 2: Rotations

$X^1 \circlearrowleft X^i$:

$$L = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & \cos\alpha & 0 & \dots & 0 & -\sin\alpha & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 \\ 0 & \sin\alpha & 0 & \dots & 0 & \cos\alpha & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \end{pmatrix} \leftrightarrow \begin{matrix} M = \\ \begin{pmatrix} \exp\left(e_i \frac{\alpha}{2}\right) & 0 \\ 0 & \exp\left(-e_i \frac{\alpha}{2}\right) \end{pmatrix} \end{matrix}$$

$X^i \circlearrowleft X^n$:

$$L = \begin{pmatrix} 1 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & \cos\alpha & 0 & \dots & 0 & -\sin\alpha \\ 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & \dots & 0 & \sin\alpha & 0 & \dots & 0 & \cos\alpha \end{pmatrix} \leftrightarrow M = \begin{pmatrix} \cos\left(\frac{\alpha}{2}\right) & e_i \sin\left(\frac{\alpha}{2}\right) \\ e_i \sin\left(\frac{\alpha}{2}\right) & \cos\left(\frac{\alpha}{2}\right) \end{pmatrix}$$

$X^1 \circlearrowleft X^n$:

$$L = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & \cos\alpha & 0 & \dots & 0 & -\sin\alpha \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & \sin\alpha & 0 & \dots & 0 & \cos\alpha \end{pmatrix} \leftrightarrow M = \begin{pmatrix} \cos\left(\frac{\alpha}{2}\right) & \sin\left(\frac{\alpha}{2}\right) \\ -\sin\left(\frac{\alpha}{2}\right) & \cos\left(\frac{\alpha}{2}\right) \end{pmatrix}$$

Category 3: Additional Transverse Rotations

$X^i \circlearrowleft X^j$:

$$L = \begin{pmatrix} 1 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & \cos\alpha & 0 & \dots & 0 & -\sin\alpha & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & \sin\alpha & 0 & \dots & 0 & \cos\alpha & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

$$\begin{aligned} & M_1 = \exp\left(-\frac{\pi}{2} e_i\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \leftrightarrow \quad & \\ & M_2 = \exp\left(\frac{\pi}{2} \left(\cos\frac{\alpha}{2}\, e_i + \sin\frac{\alpha}{2}\, e_j\right)\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$