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This paper presents an exposition of the stochastic models for the Brownian motion. The results of Einstein and Wiener are presented, together with the Uhlenbeck-Ornstein process which gives a more realistic model of the Brownian motion of a particle.

Finally, applying a one-one transformation on the forward Kolmogorov equation we have shown that the Uhlenbeck-Ornstein process can be transformed into the Wiener process.

STOCHASTIC MODELS OF THE BROWNIAN MOTION

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INTRODUCTION

In a large number of applied fields we are interested in studying the development of some system which may be regarded as subject to randomly varying influences; the theory of such a system must be presented against the background of the general theory of stochastic processes.

The Brownian motion of particles suspended in liquids or gases is due to random molecular shocks, and its theory can be based on that of stochastic processes. Macroscopically, for an ensemble of particles, the variations which occur in time are like a diffusion process. The probability density function of the random variable characterizing the system will satisfy a partial differential equation of the diffusion type, and this is the basic equation of the mathematical model of the system.

Einstein and Smoluchowski developed a satisfactory theory of the Brownian motion of free particles and of particles in a field of force, respectively. Their theories are approximate and are only valid for relatively large values of the time variable. Einstein's and Smoluchowski's results can also be arrived at by considering the Brownian motion as the limiting case of a simple random walk. N. Wiener was the first to study the mathematical model of the motion rigorously, and to prove that the functions representing the displacement of the particles are continuous, with respect to the time parameter, but non-differentiable.

A more realistic theory of the Brownian motion was advanced by Uhlenbeck and Ornstein using the Langevin equation of motion. Doob gave a rigorous mathematical justification of the Uhlenbeck-Ornstein process, according to which the functions representing the velocities of the particles are continuous with probability one, but non-differentiable. Doob was also the first to realize and show how the distribution of the displacement function in this model of the Brownian motion can be derived directly from that of the velocity function.

Recently, a method has been developed for transforming a Markov process to the Wiener process. This can be achieved by performing a one-one transformation on the backward Kolmogorov equation; here we extend this result to take into consideration the forward Kolmogorov equation. The necessary and sufficient condition for the existence of a one-one transformation is satisfied by the Uhlenbeck-Ornstein model of the Brownian motion and consequently it is possible to develop a transformation transforming the Uhlenbeck-Ornstein process into the Wiener process.

CONTINUOUS MARKOV PROCESSES

Let $\{x(t),\ t \geq 0\}$ be a continuous stochastic process defined on the real line; that is, $x(t)$ is a random variable, depending on a continuous parameter t, which assumes values in the state space $\{x: -\infty < x < \infty\}$. This process is a continuous Markov process, called also a diffusion process, if whenever $t_1 < \cdots < t_n$ the conditional distribution of $x(t_n)$ for given values of $x(t_1), \cdots, x(t_{n-1})$ depends only on $x(t_{n-1})$.

Consider a one-dimensional diffusion process and let

$$F(\tau, y; t, x) = \Pr[x(t) \leq x / x(\tau) = y] \qquad t > \tau$$

denote the conditional distribution function of the transition probabilities, which must satisfy the usual conditions

$$\lim_{x \to -\infty} F(\tau, y; t, x) = 0, \quad \lim_{x \to \infty} F(\tau, y; t, x) = 1\ .$$

Assume that $F(\tau, y; t, x)$ admits a (conditional) density function

$$f(\tau, y; t, x) = \frac{\partial}{\partial x} F(\tau, y; t, x)$$

which satisfies the conditions

$$F(\tau, y; t, x) = \int_{-\infty}^{x} f(\tau, y; t, z)\,dz, \quad \int_{-\infty}^{\infty} f(\tau, y; t, x)\,dx = 1,$$

and the Chapman-Kolmogorov equation

$$f(\tau, y; t, x) = \int_{-\infty}^{\infty} f(\tau, y; s, y) f(s, y; t, x) dy \qquad (\tau < s < t) \qquad (1)$$

which expresses the Markovian property that the changes in position during the non-overlapping time intervals (τ, s) and (s, t) are independent.

The distribution function satisfies the backward Kolmogorov equation, and the probability density satisfies the forward Kolmogorov equation also known as the Fokker-Planck equation.

In order to derive these equations we assume that the probability that $|x(t) - x(\tau)| \geq \delta$, given $x(\tau) = y$, during an infinitesimal time interval $\Delta\tau$ is small compared to $\Delta\tau$, and that the first and second partial derivatives of $F(\tau, y; t, x)$ with respect to the backward state variable y

$$\frac{\partial}{\partial y} F(\tau, y; t, x), \quad \frac{\partial^2}{\partial y^2} F(\tau, y; t, x)$$

exist and are continuous functions of y.

If at some time τ, $x(\tau) = y$, then the mean and variance of the change in $x(\tau)$ during the following interval of length $\Delta\tau$ are:

$$b(\tau, y) = \lim_{\Delta\tau \to 0} \frac{1}{\Delta\tau} \int_{-\infty}^{\infty} (y-x) f(\tau, y; \tau+\Delta\tau, x) dx \tag{2}$$

and

$$a(\tau, y) = \lim_{\Delta\tau \to 0} \frac{1}{\Delta\tau} \int_{-\infty}^{\infty} (y-x)^2 f(\tau, y; \tau+\Delta\tau, x) dx > 0 . \tag{3}$$

Under these assumptions, the backward Kolmogorov equation can be derived (1, p. 130-136) as

$$\frac{\partial}{\partial\tau} F(\tau, y; t, x) + \frac{1}{2} a(\tau, y) \frac{\partial^2}{\partial y^2} F(\tau, y; t, x) + b(\tau, y) \frac{\partial}{\partial y} F(\tau, y; t, x) = 0 \tag{4}$$

Similarly, the probability density function $f(\tau, y; t, x)$ satisfies

$$\frac{\partial}{\partial\tau} f(\tau, y; t, x) + \frac{1}{2} a(\tau, y) \frac{\partial^2}{\partial y^2} f(\tau, y; t, x) + b(\tau, y) \frac{\partial}{\partial y} f(\tau, y; t, x) = 0. \tag{4a}$$

In order to derive the Fokker-Planck equation, we assume the existence of the following continuous partial derivatives:

$$\frac{\partial}{\partial t} f(\tau, y; t, x), \quad \frac{\partial^2}{\partial x^2}[a(t, x) f(\tau, y; t, x)], \quad \frac{\partial}{\partial x}[b(t, x) f(\tau, y; t, x)] ,$$

and we have:

$$\frac{\partial}{\partial t} f(\tau, y; t, x) = \frac{1}{2} \frac{\partial^2}{\partial x^2}[a(t, x) f(\tau, y; t, x)] - \frac{\partial}{\partial x}[b(t, x) f(\tau, y; t, x)] . \tag{5}$$

Feller (10) has shown that each of the Kolmogorov diffusion equations has a unique solution which is also a solution of the Chapman-Kolmogorov equation (1). Two of the most standard methods for solving equations (4) and (5) are: assumption of separability of variables, and application of the Laplace transformation.

EINSTEIN'S THEORY OF THE BROWNIAN MOTION

Small particles suspended in fluids perform erratic movements. This phenomenon, where the particles are exposed to a great number of random molecular shocks, is referred to as Brownian motion. Einstein (7) was the first to advance a satisfactory theory of this motion based on the molecular-kinetic theory of heat. His results can be summarized as follows.

Suppose that the random variable $x = x(t)$ represents the abscissa of the particle at time t, and that the only forces acting on the particle are those due to the molecules of the surrounding medium. So, we are considering the one-dimensional Brownian motion of a free particle. We assume that each single particle executes a movement which is independent of the movement of all other particles; the movements of one and the same particle during non-overlapping time intervals are considered mutually independent.

Let $f(x_0; t, x)$ be the probability density of finding the particle at time t at the position $x = x(t)$, given that it was at x_0 at time $t = 0$. We are primarily interested in the probability

$$\int_{x_1}^{x_2} f(x_0; t, x)dx$$

that at time t the particle will be between x_1 and x_2, if it were at x_0 at time $t = 0$. Einstein showed that the probability density f must satisfy the equation

$$\frac{\partial f}{\partial t} = D \frac{\partial^2 f}{\partial x^2} \tag{6}$$

which is the forward Kolmogorov equation (5) with coefficients $a(x) = 2D$ and $b(x) = 0$, where D is a certain physical constant. The conditions imposed on f are

$$f(x_0; t, x) \geq 0, \qquad \int_{-\infty}^{\infty} f(x_0; t, x) dx = 1, \qquad \lim_{t \to 0} f(x_0; t, x) = 0 \quad \text{for} \quad x \neq x_0. \tag{7}$$

The first two conditions state that f is a probability density function of x, while the third is the initial condition and expresses the certainty that $x(0) = x_0$. Equation (6) together with conditions (7) imply that (1, p. 140)

$$f(x_0; t, x) = \frac{1}{2[\pi Dt]^{1/2}} \exp\left[-\frac{(x-x_0)^2}{4Dt}\right], \tag{8}$$

and (8) is the unique solution satisfying, for $t > 0$, conditions (7).

The constant D has the value

$$D = \frac{RT}{N\eta}$$

where R is the universal gas constant, T the absolute temperature, N the Avogadro number, and η the coefficient of friction of the particle. If the particles can be looked upon as spherical, and large compared to the molecules of the medium then

$$\eta = 6\pi \rho r$$

where ρ is the coefficient of viscosity, and r the radius of the sphere.

Furthermore, Einstein found that the mean displacement square is proportional to time,

$$E[(x-x_0)^2] = 2Dt,$$

and stressed the fact that this relation is only approximate and cannot be applied for any arbitrarily small values of t. Chandrasekhar (2, p. 25-26) has tested the density (8), and the expression for the diffusion coefficient D by observation, and found that they give satisfactory agreement for values of t which are large in comparison with the intervals between successive molecular shocks.

THE WIENER PROCESS

A simple way of introducing the mathematical model of the Brownian motion consists in regarding the motion as the limiting case of an elementary random walk.

Again let the random variable $x(t)$ represent the position at time t, of a free particle moving along the x-axis and whose initial position $x(0) = x_0$ is known. Suppose that, at every instant

$$t = n\tau, \quad n = 1, 2, \cdots$$

the particle receives a shock resulting in a displacement either Δ to the right or Δ to the left. Since we are considering a free particle, we can assume that the probabilities of moving to the left or to the right are equal, and consequently each is equal to $\frac{1}{2}$. The displacement due to each particular shock is assumed to be stochastically independent of the effects of all previous shocks, and of the initial position x_0.

Consider the probability $P(n\Delta; s\tau, m\Delta)$ that the particle is at position $m\Delta$ after s steps, if at $t = 0$ it were at $n\Delta$. If r among the s steps are directed to the right, $s-r$ are directed to the left, and the total displacement is $(2r-s)\Delta$. This displacement can equal $\nu\Delta = (m-n)\Delta$ only if s and ν are either both even or both odd. So, from the binomial distribution we have

$$P(n\Delta; s\tau, m\Delta) = \binom{s}{r}\left(\frac{1}{2}\right)^s$$

$$= \begin{cases} \dfrac{1}{2^s} \dfrac{s!}{\left(\dfrac{s+|m-n|}{2}\right)!\left(\dfrac{s-|m-n|}{2}\right)!} & \text{if } |m-n| \leq s \text{ and } |m-n|+s \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

and

$$x(s\tau) - x(0) = \Delta(2r-s) .$$

Let $\Delta \to 0$ and $\tau \to 0$ as $s \to \infty$, in such a way that

$$\frac{\Delta^2}{2\tau} = D, \quad s\tau = t, \quad n\Delta \to x_0 . \tag{10}$$

This means that the time interval between consecutive shocks, as well as the displacement caused by each shock, will tend to zero. In the limit we shall obtain a random variable $x(t)$ depending continuously on t in a way that will serve as a mathematical model of the path described by the physical particle. It follows from the Laplace-De Moivre limit theorem (8, p. 173) for the binomial distribution that

$$\lim_{s\to\infty} \sum_{x_1 \leq m\Delta \leq x_2} P(n\Delta; s\tau, m\Delta) = \frac{1}{2[\pi Dt]^{1/2}} \int_{x_1}^{x_2} \exp\left[-\frac{(x-x_0)^2}{4Dt}\right] dx;$$

that is, the displacement $x(t)-x(0)$ is normally distributed with mean zero and variance $2Dt$, and is independent of $x(0)$. Hence, Einstein's fundamental result emerges as a consequence of a limit theorem. In the same way we find that if $t_1 < t_2 < \cdots < t_n$, the displacements $x(t_2) - x(t_1), \cdots, x(t_n) - x(t_{n-1})$, are mutually independent random variables, with zero means and variances $2D(t_2 - t_1), \cdots, 2D(t_n - t_{n-1})$, respectively. We have thus a temporally homogeneous differential stochastic process.

In order that a function $x(t)$ of the time t should be acceptable as a mathematical model of the path described by the physical particle, it seems essential to require that it should be a continuous function of t, however irregular. N. Wiener (14, p. 148-151; 19) was the first to study this mathematical model rigorously. It was an important result when he proved that the functions $x(t)$ of this process are continuous with probability one. This result means that $x(t)$ can be treated as representing one of a multiplicity of continuous functions of t. Probability here is formally the study of measure on certain spaces of functions.

Even though the functions $x(t)$ are almost certainly continuous, Wiener showed that almost all such functions fail to have a derivative $\dot{x}(t)$ for any value of the argument. Physically this means that the particles in this mathematical model of the Brownian motion have no well-defined velocities.

PARTICLE IN A FIELD OF FORCE

Suppose that the particle, whose Brownian motion we are studying, is in a field of force acting in the direction of the x-axis, and that the force is given by the expression $K(x)$. Smoluchowski (16) was the first to study this case, and to show that equation (6) must be replaced by

$$\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial x^2} - \frac{1}{\eta}\frac{\partial[K(x)f]}{\partial x} \qquad f = f(x_0; t, x) , \tag{11}$$

which is the Fokker-Planck equation with coefficients

$$a(x) = 2D \qquad b(x) = \frac{K(x)}{\eta} .$$

We distinguish two important cases: a field of constant force, and an elastically bound particle.

M. Kac (13, p. 372-385) arrives at Smoluchowski's results by regarding the Brownian motion of the no longer free particle as the limiting case of a random walk with a reflecting barrier at $x = 0$; that is, $P(s\tau, 0; (s+1)\tau, \Delta) = 1$. Naturally, due to the existing outside forces the probabilities of moving to the left or to the right are no longer equal.

For the case of a field of constant force, $K(x) = -4\beta D\eta$ (β is a physical constant), equation (11) becomes

$$\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial x^2} + 4\beta D\frac{\partial f}{\partial x}. \tag{12}$$

Kac (13, p. 378-379) arrives in the limit, $\Delta \to 0$, $T \to 0$, $\frac{\Delta^2}{2T} = D$, $n\Delta \to x_0$, $sT = t$, as $s \to \infty$, at the probability density

$$f(x_0; t, x) = 4\beta \exp(-4\beta x) + \exp[-2\beta(x-x_0) - 4\beta^2 Dt] \times$$

$$\frac{2}{\pi}\int_0^\infty \left(\frac{y^2}{y^2+4\beta^2}\right)\exp(-Dy^2 t)g(x,y)g(x_0,y)dy \tag{13}$$

where $$g(x,y) = \cos(xy) - \frac{2\beta}{y}\sin(xy).$$

Formula (13) is equivalent to Smoluchowski's result (16, p. 588-589) at which he arrived directly on the basis of equation (12).

In the case of an elastically bound particle, $K(x) = -\frac{\gamma x}{\eta}$ (γ is the frequency), the Fokker-Planck equation becomes

$$\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial x^2} + \gamma\frac{\partial [xf]}{\partial x}. \tag{14}$$

Kac (13, p. 384-385) gets in the limit, for the random walk with reflecting barrier, the probability density

$$f(x_0; t, x) = \left\{\frac{\gamma}{2\pi D[1-\exp(-2\gamma t)]}\right\}^{1/2} \exp\left\{-\frac{\gamma[x-x_0\exp(-\gamma t)]^2}{2D[1-\exp(-2\gamma t)]}\right\}. \tag{15}$$

Formula (15) is the fundamental solution of equation (14) (1, p. 141-142), and is exactly the result obtained by Smoluchowski (16, p. 588).

Smoluchowski's results, based on equation (11), are approximate and are valid only for large values of the coefficient of friction η.

THE UHLENBECK-ORNSTEIN THEORY OF BROWNIAN MOTION

As was mentioned above, the theories of Einstein and Smoluchowski are only approximate. A more realistic theory of the Brownian motion was advanced by Uhlenbeck and Ornstein (17, 18). In this section we will present a summary of their results.

Let the random variable $u = u(t)$ denote the velocity, at time t, of a particle in Brownian motion. The physical problem that leads to the Uhlenbeck-Ornstein theory is the determination of the probability that a free particle at time t has velocity u given that, at $t = 0$, the velocity was u_0; that is, we want to determine the probability density

$$g(u_0; t, u) = \Pr[u(t) = u/u(0) = u_0], \ -\infty < u < \infty.$$

The basis for the development of the theory has been the Langevin equation of motion

$$m\frac{du(t)}{dt} + \eta\, u(t) = F(t), \tag{16}$$

where m is the mass of the particle, and $F(t)$ represents the random molecular impacts. From the kinetic theory the following two assumptions are made on $F(t)$ (17, p. 824):

(i) The mean of $F(t)$, at given time t, over an ensemble of particles, which have started at $t = 0$ with same velocity u_0, is zero; that is,

$$E[F(t)] = 0 . \tag{17}$$

(ii) There exists a covariance between $F(t_1)$ and $F(t_2)$ only when $|t_1 - t_2|$ is very small. More explicitly, we suppose that

$$E[F(t_1)F(t_2)] = 2D\,\phi\,(t_1 - t_2), \tag{18}$$

where $\phi(x)$ has a maximum at $x = 0$.

These two assumptions are not enough and it is further postulated (18, p. 332) that the impacts corresponding to disjoint time intervals are independent and normally distributed with zero mean and variance proportional to the length of the interval; that is, $F(t)$ is a Gaussian Markov process.

The first method for solving the problem consists of calculating $E[u(t)]$ and $E[\{u(t)\}^2]$, and of using the principle of equipartition of energy, which simply can be stated as:

$$E[u^2(t+\tau)] = E[u^2(t)] . \tag{19}$$

Using the integrating factor $\exp(\beta t)$, we get from equation (16)

$$u(t) = u_0 \exp(-\beta t) + \frac{1}{m} \exp(-\beta t) \int_0^t F(\xi) \exp(\beta \xi) d\xi \ , \tag{20}$$

where $\beta = \frac{\eta}{m}$. From relation (20) and conditions (17), (18), and (19) we get for the probability density of the velocity

$$g(u_0; t, u) = \left\{\frac{1}{2\pi D\beta[1-\exp(-2\beta t)]}\right\}^{1/2} \exp\left\{-\frac{[u-u_0\exp(-\beta t)]^2}{2D\beta[1-\exp(-2\beta t)]}\right\}, \tag{21}$$

which for $t \to \infty$ is the Maxwell distribution

$$g_1(u) = \left[\frac{1}{2\pi D\beta}\right]^{1/2} \exp\left(-\frac{u^2}{2D\beta}\right) . \tag{21a}$$

It is also shown that the random variable $u - u_0\exp(-\beta t)$ follows the Gaussian distribution.

Another way of deriving the probability density (21) is by constructing the forward Kolmogorov (Fokker-Planck) equation, of which $g(u_0; t, u)$ is the fundamental solution. From the equation of motion, Uhlenbeck and Ornstein derive the coefficients of the equation for this case as

$$a(u) = 2D\beta^2 \qquad b(u) = -\beta u \ .$$

So, the Kolmogorov equation has the form

$$\frac{\partial g}{\partial t} = \beta^2 D \frac{\partial^2 g}{\partial u^2} + \beta \frac{\partial[ug]}{\partial u}, \qquad g = g(u_0; t, u)$$

and has for its fundamental solution the density (21) (1, p. 141-142).

Again suppose that the random variable $x = x(t)$ represents the abscissa of the particle at time t. Another problem is to determine the probability $f(x_0; t, x)$ that a free particle in Brownian motion is at the position $x = x(t)$ at time t, given that it started from x_0 with initial velocity u_0.

Integrating equation (20) we get:

$$x(t) - x_0 = \frac{1}{\beta}[1-\exp(-\beta t)][\, u_0 + \frac{1}{m}\int_0^t F(\xi)\exp(\beta\xi)d\xi]. \qquad (22)$$

Using this relation together with assumptions (17) and (18), Uhlenbeck and Ornstein find for the mean and variance of the displacement

$$E[(x-x_0)] = 0 \qquad E[(x-x_0)^2] = \frac{2D}{\beta}(\beta t + e^{-\beta t} - 1). \qquad (23)$$

This is the generalization, for all values of t, of Einstein's result. Indeed, for values of t large compared to β^{-1} we get $2Dt$, which is exactly Einstein's result.

Furthermore, Ornstein and Uhlenbeck showed that the random variable

$$S = x - x_0 - \frac{u_0}{\beta}(1-e^{-\beta t})$$

follows the Gaussian distribution. For the probability density $f(x_0; t, x)$ they get

$$f(x_0;t,x) = \left[\frac{\beta}{2\pi D(2\beta t+4e^{-\beta t}-e^{-2\beta t}-3)}\right]^{1/2} \exp\left\{-\frac{\beta\left[x-x_0-\frac{u_0}{\beta}(1-e^{-\beta t})\right]^2}{2D(2\beta t+4e^{-\beta t}-e^{-2\beta t}-3)}\right\}. \tag{24}$$

For large values of t, (24) becomes the probability density (8), derived by Einstein.

THE UHLENBECK-ORNSTEIN PROCESS

The theory of Brownian motion, as developed by Uhlenbeck and Ornstein, was further elaborated upon by Doob (6) and Chandrasekhar (2). In fact, Doob was the first to make a rigorous mathematical study of the process, and to realize that the distribution of the displacements in the Uhlenbeck-Ornstein process can be obtained directly from that of the velocities.

As was seen in the last section, equation (21), the conditional distribution of $u = u(t)$, given that $u(0) = u_0$, is Gaussian with mean and variance

$$E[u(t)] = u_0 e^{-\beta t} \qquad E[(u-u_0 e^{-\beta t})^2] = D\beta(1-e^{-2\beta t}).$$

When $t \to \infty$, this conditional distribution becomes the Maxwell distribution (21a) for velocities, furnishing (12, p. 6) stationary absolute probabilities for the process. Using these absolute probabilities, Wang and Uhlenbeck (18, p. 333) describe the full distribution of the $u(t)$ process as follows: for each t, $u(t)$ is a random variable with a Gaussian distribution (21a), having zero mean, and variance $D\beta$; the process is a Markov process. This last fact means that the Maxwell distribution of $u(t_i)$ for each fixed t_i, and the conditional probabilities determine the full set of probability relations of the process. That is, if $t_1 < t_2$, we have for the joint probability density of the pair $u_1 = u(t_1)$, $u_2 = u(t_2)$

$$g_2(u_1, u_2; t_1, t_2) = g_1(u_1) g(t_1, u_1; t_2, u_2) \tag{25}$$

$$= \frac{1}{2\pi D\beta [1-e^{-2\beta(t_2-t_1)}]^{1/2}} \exp\{-\frac{u_1^2+u_2^2-2u_1u_2e^{-\beta(t_2-t_1)}}{2D\beta[1-e^{-2\beta(t_2-t_1)}]}\}$$

which is the bivariate Gaussian distribution, with zero means, equal variances $D\beta$, and correlation ceofficient $\exp[-\beta(t_2-t_1)]$.

Doob (6, p. 353-354) developed and established this $u(t)$ process rigorously by proving the following fundamental

<u>Theorem:</u> Let $u(t)$ $(t \geq 0)$ be a one-parameter family of random variables determining a stochastic process with the following properties:

(i) The process is temporally homogeneous; that is, the probability densities are unaffected by translations of the t-axis.

(ii) The process is a Markov process.

(iii) If s, t are arbitrary distinct numbers, $u(s)$, $u(t)$ have a bivariate Gaussian distribution. Define m, σ^2 by

$$m = E[u(t)], \qquad \sigma^2 = E\{[u(t)-m]^2\}. \tag{26}$$

Then the given process is of the following type: there is a constant $\beta > 0$ such that, if $t_1 < \cdots < t_n$, $u(t_1), \cdots, u(t_n)$ have an n-dimensional Gaussian distribution, with common mean m and variance σ^2, and correlation coefficients determined by $E\{[u(s+t) - m][u(s)-m]\} = \sigma^2 \exp(-\beta t)$ (Uhlenbeck-Ornstein process).

So according to this theorem, the Uhlenbeck-Ornstein process is essentially determined by three fundamental properties, of which the first two have simple physical significance. If we set $m = 0$, $\sigma^2 = D\beta$, we get for the mean and variance of $u(t) - u(0)$:

$$E[u(t)-u_0] = 0, \qquad E\{[u(t) - u_0]^2\} = 2D\beta\,(1-e^{-\beta t}). \tag{27}$$

Furthermore, Doob proved that the velocity functions $u(t)$, of the Uhlenbeck-Ornstein process, are continuous with probability one, and that almost all such functions fail to have a derivative $\dot{u}(t)$ for any value of t. Physically this means that the particles do not have a well-defined acceleration. He also found a limit for the upper bound of the velocity function $u(t)$, by proving that

$$\limsup_{t \to 0} \frac{u(t)}{[2D\beta \log t]} = 1$$

with probability one.

As was mentioned above, Doob was the first to realize that the distribution of the displacements in the Uhlenbeck-Ornstein process can be obtained from that of the velocities. Again let $x(t)$ be the abscissa of a particle in Brownian motion at time t, then

$$x(t) - x(0) = \int_0^t u(\xi)d\xi \tag{28}$$

with probability one; that is, we neglect the discontinuous functions $u(t)$ with probability zero. In order to find the distribution of the displacement, Doob proceeds as follows.

Riemann integrability of $u(t)$ implies that

$$x(t) - x_0 = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{u(t_i/n)t}{n} \tag{28a}$$

with probability one. Since the n-dimensional distribution of the random variables summed is Gaussian, it follows that the sum is Gaussian. So, the distribution of $x(t) - x_0$ is also Gaussian. Suppose again that $m = 0$, $\sigma^2 = D\beta$, then using (26) we find

$$E[x(t)-x_0] = \int_0^t E[u(\xi)]\, d\xi = 0 \tag{29}$$

and

$$E\{[x(t)-x_0]^2\} = \int_0^t \int_0^t E[u(\xi)u(\xi')]d\xi\, d\xi' \tag{29a}$$

$$= D\beta \int_0^t \int_0^t e^{-\beta|\xi-\xi'|} d\xi\, d\xi'$$

$$= \frac{2D}{\beta}(\beta t + e^{-\beta t} - 1),$$

which is Uhlenbeck's and Ornstein's original result (23).

By the same kind of argument it is shown that the two-dimensional density of $x(t)-x_0$, $u(t)$ is Gaussian, with common mean zero, and variances (29a) and $D\beta$, respectively, and correlation coefficient

$$\frac{1-e^{-\beta t}}{[2(\beta t + e^{-\beta t} - 1)]^{1/2}} \;. \tag{30}$$

Ornstein and Uhlenbeck based their results on the Langevin equation(16) of motion. In the light of Doob's results, nondifferentiability of the velocity function $u(t)$, the function $u(t)$ does not satisfy equation (16). Doob surpasses this difficulty by deriving a proper stochastic analogue of the Langevin equation, taking into consideration the fact that we do not expect $\dot{u}(t)$ to exist. Here we will present a short summary of his treatment of the Langevin equation.

First define the random variable $B(t)$ by

$$B(t) = \beta[x(t)-x(0)] + u(t) - u(0). \tag{31}$$

Then $B(t)$ has for each value of t a Gaussian distribution with mean and variance

$$E[B(t)] = 0 \qquad E\{[B(t)]^2\} = 2D\beta^2 t.$$

The distribution of $B(t+s)-B(t)$ is independent of t, and if $t_1 < \cdots < t_n$, then

$$B(t_2) - B(t_1), \cdots, B(t_n) - B(t_{n-1}) \tag{32}$$

are mutually independent random variables; that is, the $B(t)$ process is the Wiener process. The following assumption is also made: the random variable $u(0)$ will be given various initial distributions, but will always be made independent of the $B(t)$ process for $t \geq 0$. This means that the random variable $u(0)$ is assumed independent of the set of random variables (32). Physically this assumption implies that the initial velocity $u(0)$ is independent of later random molecular impacts.

We write the Langevin equation (16) in the following form

$$du(t) + \beta u(t)dt = dB(t). \tag{33}$$

We shall interpret equation (33) to mean the truth, with probability one, of

$$\int_a^b f(t)du(t) + \beta \int_a^b f(t)u(t)dt = \int_a^b f(t)dB(t), \tag{34}$$

for all a and b whenever f(t) is a continuous function. Because the second member of (34) has been defined under these hypotheses (14, p. 151-157), even though the function B(t) is known not to be of bounded variation. Doob shows that equation (33) holds for the velocity function u(t) of the Uhlenbeck-Ornstein process if B(t) is defined by (31). Then we have, with probability one,

$$\int_0^t e^{\beta\xi}\, du(\xi) = -\beta \int_0^t e^{\beta\xi} u(\xi)\, d\xi + \int_0^t e^{\beta\,\xi}\, dB(\xi) \tag{35}$$

which implies that

$$u(t) = u(0)e^{-\beta t} + e^{-\beta t} \int_0^t e^{\beta\,(\xi)} dB(\xi) \tag{36}$$

for all t, with probability one. So equation (36) furnishes the complete solution of (33) under the stated conditions.

Using the assumptions on the B(t) process and equation (36), the distribution of $u(t)-u(0)e^{-\beta t}$ can be derived, and it is found to be the same as (21), first derived by Uhlenbeck and Ornstein.

By combining equations (36) and (28), we can express the displacement function x(t) of the particle in terms of the B(t) process as follows:

$$x(t) = x(0) + \frac{u(0)}{\beta}(1-e^{-\beta t}) + \frac{1}{\beta} \int_0^t [1-e^{-\beta(t-\xi)}]\, dB(\xi), \tag{37}$$

from which the distribution of $x(t) - x_0$ can be derived.

These results can be extended to the case of particles performing the Brownian motion under the influence of an outside force $K(x)$. The Langevin equation of motion for this case is

$$\frac{du(t)}{dt} = -\beta u(t) + \frac{1}{m} K(x) + \frac{1}{m} F(t),$$

which according to Doob's treatment becomes

$$du(t) = -\beta u(t)dt + \frac{1}{m} K(x)dt + dB(t).$$

From this last equation the distribution of $u(t)$, and hence that of $x(t)$, can be obtained.

TRANSFORMATION OF THE UHLENBECK-ORNSTEIN PROCESS TO THE WIENER PROCESS

It was mentioned at the beginning that, if $x = x(t)$ is a continuous one-dimensional Markov (diffusion) process with a conditional distribution function $F(\tau, y; t, x)$, then the function F and its conditional density function

$$f(\tau, y; t, x) = \frac{\partial}{\partial x}[F(\tau, y; t, x)]$$

satisfy certain conditions, and the process is described by the backward Kolmogorov equation

$$\frac{\partial f}{\partial \tau} + \frac{a(\tau, y)}{2} \frac{\partial^2 f}{\partial y^2} + b(\tau, y) \frac{\partial f}{\partial y} = 0 , \tag{4a}$$

where the coefficients are given by relations (2) and (3).

Cherkasov (3) and Shirkov (15) consider the problem of transforming the diffusion process to a Wiener process; that is, transforming a one-dimensional continuous Markov process to a Gaussian temporally homogeneous differential process. To this end we transform the arguments and the function f itself by the following formulas

$$\begin{cases} \tau' = \phi(\tau), \quad y' = \psi(\tau, y), \quad t' = \phi(t), \quad x' = \psi(t, x) \\ f(\tau, y; t, x) = \dfrac{\partial \psi(t, x)}{\partial x} f_1(\tau', y'; t', x'), \end{cases} \tag{38}$$

first proposed by Kolmogorov. Then equation (4a) changes to an analogous equation with coefficients

$$\left\{\begin{aligned} \bar{a}(\tau, y) &= \frac{1}{2\phi'(\tau)}\, a(\tau, y)\left(\frac{\partial\psi}{\partial y}\right)^2 \\ \bar{b}(\tau, y) &= \frac{1}{\phi'(\tau)}\left[\frac{1}{2}\, a(\tau, y)\, \frac{\partial^2\psi}{\partial y^2} + b(\tau, y)\frac{\partial\psi}{\partial y} + \frac{\partial\psi}{\partial \tau}\right]. \end{aligned}\right. \tag{39}$$

We assume that equation (4a) goes over into

$$\frac{\partial f_1}{\partial \tau'} + \frac{\partial^2 f_1}{\partial y'^2} = 0; \qquad f_1 = f_1(\tau', y'; t', x') \tag{40}$$

that is, $\bar{a}(\tau, y) = 1$ and $\bar{b}(\tau, y) = 0$. It is known that the solution of equation (40), which satisfies the usual regularity conditions of the density functions and the Chapman-Kolmogorov equation (1), is

$$f_1(\tau', y'; t', x') = \frac{1}{2[\pi(t' - \tau')]^{1/2}}\; e^{-\frac{(x'-y')^2}{4(t'-\tau')}} \tag{41}$$

giving the Wiener process. From this it is easy to obtain the function $f(\tau, y; t, x)$ which is the solution of equation (4a), provided that transformation (38) is one-one. Indeed, Cherkasov showed that under certain conditions there exists a one-one transformation which takes equation (4a) into equation (40).

Here we propose to apply the same treatment to the forward

Kolmogorov (Fokker-Planck) equation

$$\frac{\partial f}{\partial t} = \frac{1}{2} \frac{\partial^2[a(t,x)f]}{\partial x^2} - \frac{\partial[b(t,x)f]}{\partial x}, \tag{5}$$

which can also be written as follows

$$\frac{\partial f}{\partial t} = \frac{1}{2} a(t,x) \frac{\partial^2 f}{\partial x^2} + \left[\frac{\partial a(t,x)}{\partial x} - b(t,x)\right] \frac{\partial f}{\partial x} + \left[\frac{1}{2} \frac{\partial^2 a(t,x)}{\partial x^2} - \frac{\partial b(t,x)}{\partial x}\right] f. \tag{42}$$

Applying transformation (38) to equation (42), it changes to an analogous equation with coefficients

$$\left\{ \begin{aligned} \bar{a}(t,x) &= \frac{a(t,x)}{\phi'(t)} \left(\frac{\partial \psi}{\partial x}\right)^2 \\ \bar{b}(t,x) &= \frac{1}{\phi'(t)} \left\{ \left[b(t,x) - \frac{\partial a(t,x)}{\partial x}\right] \frac{\partial \psi}{\partial x} - \frac{3}{2} a(t,x) \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial \psi}{\partial t} \right\}. \end{aligned} \right. \tag{43}$$

We assume that equation (42) goes over into

$$\frac{\partial f_1}{\partial t'} = \frac{1}{2} \frac{\partial^2 f_1}{\partial x'^2}, \tag{44}$$

whose solution is

$$f_1(\tau', y'; t', x') = \frac{1}{[2\pi(t'-\tau')]^{1/2}} e^{-\frac{(x'-y')^2}{2(t'-\tau')}} \tag{44a}$$

giving the Wiener process. Using Cherkasov's method (3, p. 374-377) we prove the following

<u>Theorem.</u> Let

$$\begin{cases} \alpha(t,x) = \sqrt{a}(t,x), \qquad \beta(t,x) = \alpha(t,x)\displaystyle\int_0^x \frac{d\xi}{\alpha(t,\xi)} \\ \\ \gamma(t,x) = 2b(t,x) - \frac{1}{2}a_x(t,x) - \alpha(t,x)\displaystyle\int_0^x a_t(t,\xi)[\alpha(t,\xi)]^{-3}d\xi \,, \end{cases} \tag{45}$$

and suppose that there exist continuous derivatives α_{xx}, β_{xx}, γ_{xx} for all real values of x, and that the functions α, $\frac{1}{\alpha}$ are bounded. Then there exists a one-one transformation of the type (38) which transforms equation (5) into (44) if and only if the Wronskian of α, β, γ is identically zero. If this condition is satisfied, the desired transformation is given by the following formulas

$$\phi(t) = \int_{t_0}^{t} \exp\left[-\int_{t_0}^{\xi} \frac{W(s,x)}{\alpha(s,x)} ds\right] dx \tag{46a}$$

and

$$\psi(t,x) = \frac{\beta(t,x)}{\alpha(t,x)} \exp\left[-\frac{1}{2}\int_{t_0}^{t} \frac{W(s,x)}{\alpha(s,x)} ds\right] + \frac{1}{2}\int_{t_0}^{t} \frac{P(\xi,x)}{\alpha(\xi,x)} \exp\left[-\frac{1}{2}\int_{t_0}^{\xi} \frac{W(s,x)}{\alpha(s,x)} ds\right] d\xi \tag{46b}$$

where $W(t, x)$ is the Wronskian of α and γ, and $P(t, x)$ is the Wronskian of β and γ.

Proof: Necessity. Syppose that after applying transformation (38) to equation (5) we get

$$\bar{a}(t, x) = 1 \qquad \bar{b}(t, x) = 0.$$

The first equation of (43) gives

$$\frac{\partial \psi}{\partial x} = \left[\frac{\phi'(t)}{a(t, x)}\right]^{1/2}, \qquad \frac{\partial^2 \psi}{\partial x^2} = -\frac{[\phi'(t)]^{1/2}}{2[a(t, x)]^{3/2}} a_x(t, x).$$

Integrating the first of the above equations we get

$$\psi(t, x) = [\phi'(t)]^{1/2} \int_0^x \frac{d\xi}{a(t, \xi)} + \phi_1(t),$$

where $\phi_1(t)$ is assumed to have continuous first derivative. From the expression for $\psi(t, x)$ we find

$$\frac{\partial \psi}{\partial t} = \frac{\phi''(t)}{2[\phi'(t)]^{1/2}} \int_0^x \frac{d\xi}{a(t, \xi)} - \frac{[\phi'(t)]^{1/2}}{2} \int_0^x a_t(t, \xi)[a(t, \xi)]^{-3} d\xi + \phi_1'(t).$$

Substituting $\frac{\partial \psi}{\partial t}$, $\frac{\partial \psi}{\partial x}$, and $\frac{\partial^2 \psi}{\partial x^2}$ into the second of equations (43) we obtain

$$b(t,x)\left[\frac{\phi'(t)}{a(t,x)}\right]^{1/2} - \frac{1}{4}\left[\frac{\phi'(t)}{a(t,x)}\right]^{1/2} a_x(t,x) + \frac{\phi''(t)}{2[\phi'(t)]^{1/2}} \int_0^x \frac{d\xi}{a(t,\xi)} -$$

$$- \frac{[\phi'(t)]^{1/2}}{2} \int_0^x a_t(t,\xi)[a(t,\xi)]^{-3} d\xi + \phi_1'(t) = 0.$$

Multiplying this last equation by $2\left[\frac{a(t,x)}{\phi'(t)}\right]^{1/2}$, we obtain

$$\frac{2\phi_1'(t)}{[\phi'(t)]^{1/2}}[a(t,x)]^{1/2} + \frac{\phi''(t)}{\phi'(t)}[a(t,x)]^{1/2} \int_0^x \frac{d\xi}{a(t,\xi)} + \{2b(t,x) - \frac{1}{2}a_x(t,x) -$$

$$- [a(t,x)]^{1/2} \int_0^x a_t(t,\xi)[a(t,\xi)]^{-3} d\xi\} = 0 \ ;$$

which according to the introduced notation can be written as

$$M(t)\alpha(t,x) + N(t)\beta(t,x) + \gamma(t,x) = 0, \tag{47}$$

where

$$M(t) = 2\phi_1'(t)[\phi'(t)]^{-1/2} \quad \text{and} \quad N(t) = \phi''(t)[\phi'(t)]^{-1}.$$

If equation (47) is differentiated twice with respect to x, and the resulting system of three equations is considered, then the condition of the theorem is obtained.

Sufficiency. If the condition

$$\Delta = \begin{vmatrix} \alpha(t,x) & \beta(t,x) & \gamma(t,x) \\ \alpha_x(t,x) & \beta_x(t,x) & \gamma_x(t,x) \\ \alpha_{xx}(t,x) & \beta_{xx}(t,x) & \gamma_{xx}(t,x) \end{vmatrix} \equiv 0 \qquad (48)$$

is fulfilled, we show following Cherkasov's method that formulas (46) transform equation (5) into equation (44). To this end it is easy to show (15, p. 157) that the expressions

$$\frac{W(t,x)}{\alpha(t,x)} \quad \text{and} \quad \frac{P(t,x)}{\alpha(t,x)}$$

are independent of x.

Now using relations (43) and (46) we find

$$\bar{a}(t,x) = \frac{a(t,x)}{\exp[-\int_{t_0}^{t} \frac{W(s,x)}{\alpha(s,x)}\, ds]} \left\{\frac{1}{\alpha(t,x)} \exp[-\frac{1}{2}\int_{t_0}^{t} \frac{W(s,x)}{\alpha(s,x)} ds]\right\}^2 = 1. \qquad (49)$$

Furthermore,

$$\bar{b}(t,x) = \exp[\frac{1}{2}\int_{t_0}^{t} \frac{W(s,x)}{\alpha(s,x)} ds] \left\{\frac{b(t,x)}{\alpha(t,x)} - \frac{a_x(t,x)}{4\alpha(t,x)} - \frac{W(t,x)}{2\alpha(t,x)} \int_0^x \frac{d\xi}{\alpha(t,\xi)} - \right.$$

$$\left. - \frac{1}{2} \int_0^x a_t(t,\xi)[\alpha(t,\xi)]^{-3} d\xi + \frac{P(t,x)}{2\alpha(t,x)} \right\},$$

which using our notation can be written as

$$\overline{b}(t,x) = \frac{\exp\left[\frac{1}{2}\int_{t_0}^{t} \frac{W(s,x)}{\alpha(s,x)} ds\right]}{2\alpha(t,x)} \left[\gamma(t,x) - \frac{W(t,x)}{\alpha(t,x)}\beta(t,x) + P(t,x)\right]. \tag{50}$$

It can easily be shown that

$$P(t,x) = \frac{W(t,x)}{\alpha(t,x)}\beta(t,x) - \gamma(t,x);$$

from this it follows that $\overline{b}(t,x) = 0$.

Using the above theorem we propose to show that the Uhlenbeck-Ornstein process can be transformed into the Wiener process. Indeed, it was found (17) that the conditional density function $g(\tau, y; t, u)$ of the velocity $u = u(t)$ of a free particle in Brownian motion satisfies the Fokker-Planck equation (5) with coefficients

$$a(t,u) = 2D\beta^2, \quad b(t,u) = -\beta u.$$

Using our notation (45), we have in this case

$$\alpha(t,u) = \sqrt{2D}\beta, \quad \beta(t,u) = u, \quad \gamma(t,u) = -2\beta u; \tag{51}$$

whose Wronskian is

$$\Delta = \begin{vmatrix} \sqrt{2D}\beta & u & -2\beta u \\ 0 & 1 & -2\beta \\ 0 & 0 & 0 \end{vmatrix} \equiv 0.$$

So according to our theorem, there exists a one-one transformation of the type (38) transforming the Uhlenbeck-Ornstein process into the Wiener process. The transformation is given by the relations

$$\phi(t) = \frac{1}{2\beta}\,(e^{2\beta t}-1), \quad \psi(t,u) = \frac{ue^{\beta t}}{\sqrt{2D}\beta} \;. \tag{52}$$

Using equation (44a), and the last of equations (38), we have for the conditional density of the velocity

$$g(\tau, y; t, u) = \frac{e^{\beta t}}{\sqrt{2D}\beta}\left[\frac{\beta}{\pi(e^{2\beta t}-e^{2\beta\tau})}\right]^{1/2} \exp\left[-\frac{(ue^{\beta t}-ye^{\beta\tau})^2}{2D\beta(e^{2\beta t}-e^{2\beta\tau})}\right];$$

which can be rewritten as

$$g(\tau, y; t, u) = \left[\frac{e^{-2\beta\tau}}{2\pi D\beta(e^{-2\beta\tau}-e^{-2\beta t})}\right]^{1/2} \exp\left[-\frac{(ue^{-\beta\tau}-ye^{-\beta t})^2}{2D\beta(e^{-2\beta\tau}-e^{-2\beta t})}\right]. \tag{53}$$

Setting $\tau = 0$, $u(0) = u_0$, we obtain

$$g(u_0; t, u) = \left[\frac{1}{2\pi D\beta(1-e^{-2\beta t})}\right]^{1/2} \exp\left[-\frac{(u-u_0e^{-\beta t})^2}{2D\beta(1-e^{-2\beta t})}\right],$$

which is exactly the result (21) obtained by Uhlenbeck and Ornstein. As was mentioned in the last section, from the above distribution one can obtain the distribution of the displacement function of the Uhlenbeck-Ornstein process.

So, a complete description of the Uhlenbeck-Ornstein process can be obtained directly from the Wiener process (44a) and transformation (38), according to formulas (52).

SUMMARY

The theory of stochastic processes has been systematically developed and has been applied to a wide variety of problems in different fields. One such problem is the Brownian motion of particles suspended in liquids or gases.

The theory of Brownian motion has been studied extensively both from purely physical considerations and against the background of Markov processes.

Einstein and Smoluchowski developed an approximate theory of the motion. Their model was treated rigorously by Wiener who proved that the displacement function of the particle is continuous and non-differentiable with probability one.

Uhlenbeck and Ornstein advanced a more realistic model of the Brownian motion, which was rigorously treated by Doob. Based on Wiener's results, Doob showed that the particles of this model have a well-defined velocity but not a well-defined acceleration.

In both of the above models, the Kolmogorov equation, satisfied by the probability density function of the random variable characterizing the model, is the basic equation. Applying a one-one transformation on the forward Kolmogorov equation, we have shown in this work that the Uhlenbeck-Ornstein process can be transformed into the Wiener process. This means that, using a simple transformation we

can transform a continuous Markov process to a Gaussian temporally homogeneous differential process.

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