AN EXTENSION OF WIDDER'S THEOREM

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TABLE OF CONTENTS

AN EXTENSION OF WIDDER'S THEOREM

INTRODUCTION

In 1944 D.V. Widder proved that any positive solution u of the heat equation $u_{xx} = u_t$ in a strip $S: \{0 < t \leq T\}$ is representable in the form of a Stieltjes integral

$$u(x,t) = \int_{-\infty}^{\infty} k(x-y,t)\, d\alpha(y),$$

where $k(x,t) = (4\pi t)^{-1/2}\, e^{-\frac{x^2}{4t}}$, and where α is a nondecreasing function.

The argument depends essentially on the maximum principle, or equivalently, on the fact that $k(x,t)$ is positive. It seems therefore reasonable to expect that a similar theorem would hold for more general parabolic equations, since their solutions also satisfy the maximum principle. The theorem presented here is more directly analogous to a simplified version of Widder's theorem due to P. Ungar (See F. John [5]). The equation considered is $Lu = u_{xx} + cu - u_t = 0$, where $c = c(x,t)$ is bounded and uniformly Hölder continuous. The result is that if u is a positive solution of $Lu = 0$ in $t_0 < t \leqq T$, and is continuous in $t_0 \leqq t \leqq T$, then

$$u(x,t) = \int_{-\infty}^{\infty} \Gamma(x,\xi;\ t,t_0)\ u(\xi,t_0)d\xi,$$

where Γ is the fundamental solution of $Lu = 0$ with singularity at (ξ,t_0).

PART I

REDUCTION OF $a(x,t)u_{xx} + b(x,t)u_x + c(x,t)u = u_t$ TO $u_{xx} + c(x,t)u = u_t$

Consider the partial differential equation

1) $a(x,t)u_{xx} + b(x,t)u_x + c(x,t)u = u_t$,

where $M \geqq a(x,t) \geqq a_0 > 0$, and $a(x,t) \varepsilon c'$.

Let $\tau = t$, $\xi = \int_0^x \frac{dx}{\sqrt{a(x,t)}}$

$\xi_x = \frac{1}{\sqrt{a(x,t)}}$ $\qquad \xi_{xx} = -1/2[a(x,t)]^{-3/2}a_x$

$\xi_x^{\ 2} = \frac{1}{a(x,t)}$ $\qquad a\xi_{xx} = -1/2\ \frac{a_x}{\sqrt{a}}$

$\therefore\ a(x,t)\xi_x^{\ 2} = 1$ $\qquad \xi_t = -\int_0^x 1/2\ a_t \cdot a^{-3/2}dx$

$u_t = u_\tau + u_\xi \xi_t$

$u_x = u_\xi \xi_x$

$u_{xx} = u_{\xi\xi}\xi_x^{\ 2} + u_\xi \xi_{xx}$

Substituting these into (1), we obtain

$u_\tau + u_\xi \xi_t = a(u_{\xi\xi}\xi_x^{\ 2} + u_\xi \xi_{xx}) + bu_\xi \xi_x + cu$

$u_\tau = u_{\xi\xi} + u_\xi(a\xi_{xx} + b\xi_x - \xi_t) + cu.$

Let $a\xi_{xx} + b\xi_x - \xi_t \equiv b'$.

Thus we obtain for (1),

2) $u_\tau = u_{\xi\xi} + b'u_\xi + cu$.

Let $u = fv$.

We wish to determine f so that we can eliminate the u_ξ term in (2).

$u = fv$

$u_\xi = fv_\xi + vf_\xi$

$u_{\xi\xi} = fv_{\xi\xi} + 2f_\xi v_\xi + vf_{\xi\xi}$

$u_\tau = v_\tau f + vf_\tau$

Thus (2) becomes

$v_\tau f + vf_\tau = fv_{\xi\xi} + 2f_\xi v_\xi + vf_{\xi\xi} + b'v_\xi + b'vf_\xi + cfv.$

$v_\tau f = v_{\xi\xi} f + v_\xi(2f_\xi + b'f) + v(cf + b'f_\xi + f_{\xi\xi} - f_\tau).$

From this we may determine f.

Set $2f_\xi + b'f = 0$

$$\frac{f_\xi}{f} = -1/2b'$$

$$\log f = -1/2 \int_0^\xi b'd\xi$$

$$f = e^{-1/2 \int_0^\xi b'd\xi}$$

After determining f in this manner, (2) reduces to

3) $v_\tau = v_{\xi\xi} + c'(\xi,\tau)v,$

where $c' = c + b'\frac{f_\xi}{f} + \frac{f_{\xi\xi}}{f} - \frac{f_\tau}{f}$.

We now demonstrate the one-to-one character of the above mappings. We wish to show first that the mapping $(x,t) \to (\xi,\tau)$ is one-to-one. Assume $(x_1,t_1) \to (\xi_1,\tau_1)$ and $(x_2,t_2) \to (\xi_2,\tau_2)$. We must show that if $(\xi_1,\tau_1) = (\xi_2,\tau_2)$, then $(x_1,t_1) = (x_2,t_2)$. Thus, we assume

$$\int_0^{x_1} \frac{ds}{\sqrt{a(s,t_1)}} = \int_0^{x_2} \frac{ds}{\sqrt{a(s,t_2)}}$$

But $\tau_1 = \tau_2$ implies already that $t_1 = t_2$.

So we have $\displaystyle\int_0^{x_1} \frac{ds}{\sqrt{a(s,t_1)}} = \int_0^{x_2} \frac{ds}{\sqrt{a(s,t_2)}}$ or

$$\int_{x_2}^{x_1} \frac{ds}{\sqrt{a(s,t_1)}} = 0.$$

We now have three cases to consider.

Case I. $x_1 = x_2$. But in this case, we have already exactly what we wanted to show.

Case II. $x_1 > x_2$. Then

$$0 = \int_{x_2}^{x_1} \frac{ds}{\sqrt{a(s,t_1)}} \geqq \int_{x_2}^{x_1} \frac{ds}{M} = \frac{1}{M}(x_1 - x_2) \geqq 0,$$

which implies that $x_1 = x_2$.

Case III. $x_2 > x_1$. Then

$$0 = \left| \int_{x_2}^{x_1} \frac{ds}{\sqrt{a(s,t_1)}} \right| = \int_{x_1}^{x_2} \frac{ds}{\sqrt{a(s,t)}}$$

$$\geqq \int_{x_1}^{x_2} \frac{ds}{M} = \frac{1}{M}(x_2 - x_1) \geqq 0,$$

which again implies that $x_1 = x_2$.

To see that the mapping $u \to fv$ is one-to-one, we need only note that $v = \frac{u}{f}$ gives the reverse transformation, since f is never zero.

Thus it is sufficient to consider solutions of (3). We drop, now, the c' notation and write finally

4) $u_t = u_{xx} + c(x,t)u.$

PART II

FORMULATION OF THE PROBLEM AND PRELIMINARY THEOREMS

Let $L \equiv \frac{\partial^2}{\partial x^2} + c(x,t) - \frac{\partial}{\partial t}$.

We seek conditions for positive solutions to the initial value problem

5) $Lu = 0$

$u(x,t_0) = \varphi(x)$

Both Dressel (see [1] and [2]) and Feller (see [3]) have proven the existence of a fundamental solution $\Gamma(x,\xi;\ t,t_0)$ to $Lu = 0$, i.e. a solution which satisfies the following conditions:

i) $L(\Gamma(x,\xi;\ t,t_0)) = 0$ for $t > t_0$,

ii) $\lim\limits_{t\to t_0} \int_{-\infty}^{\infty} \Gamma(x,\xi;\ t,t_0)\ \varphi(\xi)d\xi = \varphi(x)$,

provided φ is continuous and bounded in $(-\infty, \infty)$.

Before proceeding any further, we shall need two preliminary results, the second of which is to be found in Feller (see [3]).

Theorem 1. $\int_{-\infty}^{\infty} k(x-\xi, t-\tau)k(\xi-y, \tau-t_0)d\xi = k(x-y, t-t_0)$

where $k(x,t) = (4\pi t)^{-1/2} e^{-\frac{x^2}{4t}}$

This theorem may be established by direct computation.

Theorem 2. $\int_{-\infty}^{\infty} \Gamma(x,\xi;\ t,\tau)\ \Gamma(\xi,y;\ \tau,t_0)d\xi = \Gamma(x,y;\ t,t_0)$

where Γ is the fundamental solution of $Lu = 0$.

We shall use Dressel's representation of Γ in order to be able to estimate it properly. In order to do so, however, and in order to be able to apply Dressel's theorems, we must at this point make the following assumptions on $c(x,t)$:

i) $|c(x,t)| \leqq M$

ii) $c(x,t)$ satisfies a Hölder condition of order, γ, $0 < \gamma \leqq 1$, of the type $|c(x_1,t_1)-c(x_0,t_0)| \leqq N[|x_1-x_0|^\gamma + |t_1-t_0|^\gamma]$, where N is a constant.

We write $\Gamma(x,\xi;\ t,\ t_0)$ in the form

$$6)\quad \Gamma(x,\xi;\ t,t_0) = k(x-\xi,t-t_0)$$

$$+ \int_{t_0}^{t} \int_{-\infty}^{\infty} k(x-s,t-\tau)\ f(s,\tau;\ \xi,t_0)\,ds\,d\tau.$$

We now apply the operator L to (6). Since $L(\Gamma) = 0$ by hypothesis, we obtain, upon applying Dressel's theorems 1 and 2, the following integral equation for f.

$$7)\quad f(x,t;\ \xi,t_0) = c(x,t)\ k(x-\xi,t-t_0)$$

$$+ c(x,t) \int_{t_0}^{t} \int_{-\infty}^{\infty} k(x-s,t-\tau)\ f(s,\tau;\ \xi,t_0)\,ds\,d\tau.$$

Thus $|f(x,t;\ \xi,t_0)| \leqq Mk(x-\xi,t-t_0)$

$$+ M \int_{t_0}^{t} \int_{-\infty}^{\infty} k(x-s,t-\tau)\,|f(s,\tau;\ \xi,t_0)|\,ds\,d\tau.$$

Iterating and applying theorem 1, we obtain the series

$$|f| \leqq Mk + M^2(t-t_0)k + \frac{(t-t_0)^2}{2!} M^3 k + \ldots + \frac{(t-t_0)^n}{n!} M^{n+1} k + \ldots$$

$$= Mk[1 + (t-t_0)\ M + \frac{(t-t_0)^2}{2!} M^2 + \ldots + \frac{(t-t_0)^n}{n!} M^n + \ldots]$$

$$= Mk(x-\xi,t-t_0)e^{M(t-t_0)}.$$

Thus we have the following estimate for Γ:

$$|\Gamma(x,\xi;\ t,t_0)| \leqq k(x-\xi,t-t_0)$$

$$+ M\int_{t_0}^{t}\int_{-\infty}^{\infty} k(x-s,t-\tau)\ k(s-\xi,\tau-t_0)e^{M(\tau-t_0)}\,ds\,d\tau$$

$$\leqq k(x-\xi,t-t_0) + Me^{M(t-t_0)}\int_{t_0}^{t} k(x-\xi,t-t_0)d\tau$$

$$= k(x-\xi,t-t_0)[1 + M(t-t_0)e^{M(t-t_0)}].$$

Now write

8) $\Gamma(x,\xi,t,t_0) = k(x-\xi,t-t_0) + R(x,\xi;\ t,t_0)$,

where $|R(x,\xi;\ t,t_0)| \leqq Me^{M(t-t_0)}(t-t_0)\ k(x-\xi,t-t_0)$.

We are now in a position to prove the following theorem.

Theorem 3. If $|\varphi(x)| \leqq Ke^{\alpha x^2}$, where K and α are constants, and if φ is integrable on every finite interval, then

9) $u(x,t) = \int_{-\infty}^{\infty} \Gamma(x,\xi;\ t,t_0)\ \varphi(\xi)d\xi$

is a solution to $Lu = 0$ in S: $\{0 < t \leqq T\}$ provided $c(x,t)$ satisfies the above conditions. Furthermore, if φ is continuous at x_0, then

$$10) \quad \lim_{(x,t)\to(x_0,t_0)} u(x,t) = \varphi(x_0)$$

Proof. Under the same conditions as in this theorem, we know from the corresponding theorem on the heat equation (see e.g. F. John [5] or G. Hellwig [6]) that

$$11) \quad \lim_{(x,t)\to(x_0,t_0)} \int_{-\infty}^{\infty} k(x-\xi,t-t_0)\varphi(\xi)d\xi = \varphi(x_0).$$

Furthermore, by the estimate that we derived for $R(x,\xi;\ t,t_0)$, we have

$$\lim_{(x,t)\to(x_0,t_0)} \int_{-\infty}^{\infty} R(x,\xi;\ t,t_0)\varphi(\xi)d\xi = 0.$$

This proves (10). To prove (9), we note that

$$Lu = L\int_{-\infty}^{\infty} \Gamma(x,\xi;\ t,t_0)\varphi(\xi)d\xi = \int_{-\infty}^{\infty} L\Gamma(x,\xi;\ t,t_0)\varphi(\xi)d\xi = 0,$$

provided, of course, that the operations of integration and differentiation may be interchanged. But this is a standard argument similar to that for the heat equation (again see [5] or [6]). This proves the theorem.

For the work to follow, we shall have need of two preliminary results.

Lemma 1. If $|c(x,t)| \leqq M$ in the equation $Lu = 0$, we may assume without loss of generality that $c(x,t) \leqq 0$.

Proof. Consider $Lu = 0$ with $|c| \leqq M$. Set $u = e^{Mt}v$. Then $u_{xx} = e^{Mt}v$ and $u_t = Me^{Mt}v + e^{Mt}v_t$.

Thus $Lu = 0$ becomes

$$e^{Mt}v_{xx} + ce^{Mt}v = Me^{Mt}v + e^{Mt}v_t.$$

12) $v_{xx} + [M - c(x,t)]v = v_t.$

$c'(x,t) \equiv M - c(x,t) \leqq 0.$

Thus we have to consider

13) $v_{xx} + c'(x,t)v = v_t$

where $-2M \leqq c'(x,t) \leqq 0.$

Lemma 2. $\Gamma(x,\xi;\ t,t_0) \geqq 0.$

Proof. $\Gamma(x,\xi;\ t,t_0) = k(x-\xi,t-t_0)$

$$+ \int_{t_0}^{t} \int_{-\infty}^{\infty} k(x-s,t-\tau)f(s,\tau;\ \xi,t_0)dsd\tau.$$

We now proceed as before, but this time we estimate f from below.

$$f(x,t;\ \xi,t_0) \geqq -Mk\ -M^3k\frac{(t-t_0)^2}{2!}\ -M^5k\ \frac{(t-t_0)^4}{4!}\ -\dots$$

$$= -Mk\ [1 + M^2\ \frac{(t-t_0)^2}{2!} + \frac{M^4(t-t_0)^4}{4!} + \dots]$$

$$= -Mk\ \cosh M(t-t_0),$$

where of course $k = k(x-\xi,t-t_0)$.

14) $\Gamma(x,\xi;\ t,t_0) \geqq k(x-\xi,t-t_0)$

$$-\left|\int_{t_0}^{t}\int_{-\infty}^{\infty} k(x-s,t-\tau)k(s-\xi,\tau-t_0)\cosh M(\tau-t_0)dsd\tau\right|$$

$$= k(x-\xi,t-t_0)\ -\left|M\int_{t_0}^{t} k(x-\xi,t-t_0)\ \cosh M(\tau-t_0)d\tau\right|$$

$$= k(x-\xi,t-t_0)\ [1\ -\ \sinh M(t-t_0)].$$

Now for t sufficiently close to t_0, $\sinh M(t-t_0) < \varepsilon$. So at least for a small strip we know that Γ is positive. Say $\Gamma(x,\xi;\ t,t_0) \geqq 0$ if $|t-t_0| \leqq \beta$, since we know that such a β exists.

By theorem 2

$$\Gamma(x,\xi;\ t,t_0) = \int_{-\infty}^{\infty} \Gamma(x,y;\ t,\tau)\Gamma(y,\xi;\ \tau,t_0)dy,$$

where $\Gamma(x,y;\ t,\tau) \geqq 0$ if $0 \leqq t-\tau \leqq \beta$ and

$\Gamma(y,\xi;\ \tau,t_0) \geqq 0$ if $0 \leqq \tau-t_0 \leqq \beta$.

This implies that $0 \leqq t-t_0 \leqq 2\beta$.

Given any T: $\beta < T \leqq 2\beta$, choose $\tau-t_0 = \frac{T}{2}$ and

$t-\tau = \frac{T}{2}$.

Now apply theorem 2,

$\Gamma = \int \Gamma\Gamma > 0$.

$\therefore\ \Gamma(x,\xi;\ t,t_0) \geqq 0$ if $\beta < t-t_0 \leqq 2\beta$.

By continuing this process, we see finally that for any $t > t_0$, $\Gamma \geqq 0$.

Recall now that $\Gamma(x,\xi;\ t,t_0) = k(x-\xi,t-t_0) + R(x,\xi;\ t,t_0)$, where $|R(x,\xi;\ t,t_0)| \leqq Me^{M(t-t_0)}k(x-\xi,t-t_0)$.

Lemma 3. There exists a τ such that $1/2k \leqq \Gamma \leqq 3/2k$ for $0 < t-t_0 \leqq \tau$.

Proof. Choose τ so small that $M\tau e^{M\tau} < 1/2$.

Then $\Gamma \leqq k + M\tau e^{M\tau} k \leqq (1 + 1/2)k$

and $\Gamma \geqq k - M\tau e^{M\tau} k \geqq (1 - 1/2)k$.

Lemma 4. Let T be given. Then there exists an n such that

$(1/2)^n k \leqq \Gamma \leqq (3/2)^n k$ for $0 < t-t_0 \leqq T$.

Proof. A) Let $\tau < t-t_0 \leqq 2\tau$.

Then $\Gamma(x,\xi;\ t,t_0)$

$$= \int_{-\infty}^{\infty} \Gamma(x,y;\ t_0 + \tau, t_0)\Gamma(y,\xi;\ t,t_0 + \tau)dy$$

$$\leqq (3/2)^2 \int_{-\infty}^{\infty} k(x-y,\tau)k(y-\xi,t-t_0-\tau)dy$$

$$= (3/2)^2 k(x-\xi,t-t_0)$$

and $\Gamma(x,\xi;\ t,t_0) \geqq (1/2)^2 k(x-\xi,t-t_0)$.

B) Complete by induction until $n\tau > T$.

We are now in a position to prove that the representation given in theorem 3 is unique. Also from this point on, we shall set, simply as a matter of convenience, $t_0 = 0$.

Consider the function

$$v = \frac{\varepsilon}{\sqrt{1 - 4A't}} \, e^{\frac{A'x^2}{\sqrt{1 - 4A't}}}$$

Observe that

$$\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial t} = 0 \quad \text{for} \quad 0 < t < 1/4A' \, ,$$

$v(x,0) = \varepsilon e^{A'x^2}$ and $v(x,0) \leqq v(x,t)$.

We consider also $v^* = \varepsilon\int_{-\infty}^{\infty} \Gamma(x,y;\, t,0)e^{A'y^2}dy.$

Now for appropriate n, $n\tau > T$,

$$v^* \leqq \varepsilon(3/2)^n \int_{-\infty}^{\infty} k(x-y,t)e^{A'y^2}dy = (3/2)^n v, \quad 0 < t \leqq T.$$

$v^* \geqq (1/2)^n v.$

Theorem 4. Let u be continuous for $0 \leqq t \leqq T$

and all x,

$Lu = 0, \; 0 < t \leqq T,$

$|u| \leqq \mu e^{Ax^2}, \; 0 \leqq t \leqq T,$

$u(x,0) = 0,$

then $u(x,t) \equiv 0$ in $0 \leqq t \leqq T.$

Proof. Let $\varepsilon > 0$, $A' > A$ be given. Let

$$a \geqq \left(\frac{\log \frac{\mu}{\varepsilon}}{A'-A}\right)^{1/2} .$$

Then $|u(\pm a,t)| \leqq \mu e^{Aa^2} \leqq \varepsilon e^{A'a^2}$.

Let D be the rectangle with corners $(-a,0)$, $(a,0)$, $(a,1/4A')$, $(-a,1/4A')$.

Then on $x = \pm a$,

$$|u(x,\pm a)| \leqq \mu e^{Aa^2} \leqq \varepsilon e^{A'a^2} \leqq v \leqq 2^n v^*.$$

Furthermore $|u(x,0) \leqq 2^n v^*$, since $u(x,0) = 0$.

Therefore, by the maximum principle, both

$2^n v^* \pm u \geqq 0$ in D.

Letting $a \to \infty$ implies $2^n v^* \pm u \geqq 0$ for $0 \leqq t \leqq 1/4A'$.

Now letting $\varepsilon \to 0$, we obtain $\pm u \geqq 0$ for $0 \leqq t \leqq 1/4A'$, that is $u = 0$ for $0 \leqq t \leqq 1/4A'$.

Letting $A' \to A$, we have $u = 0$ for $0 \leqq t < 1/4A$ and by continuity $u = 0$ for $0 \leqq t \leqq 1/4A$.

By induction this result can be extended to $0 \leqq t \leqq T$ and this proves the theorem.

PART III

EXTENSION OF WIDDER'S THEOREM

Theorem 5. If $Lu = 0$ and $u(x,t)$ is continuous for $0 \leqq t \leqq T$, and $u(x,t) \geqq 0$, with $-M \leqq c(x,t) \leqq 0$, and where c satisfies a uniform Hölder condition of order γ, then

$$u(x,t) = \int_{-\infty}^{\infty} \Gamma(x,\xi;\ t,0)\varphi(\xi)d\xi$$

where $u(x,0) = \varphi(x)$. Thus $u(x,t)$ is uniquely determined by $\varphi(x)$.

Proof. We first show that

$$15)\quad u(x,t) \geqq \int_{-\infty}^{\infty} \Gamma(x,\xi;\ t,0)\varphi(\xi)d\xi.$$

Let $\varepsilon > 0$, $A > 0$ be arbitrary.

Further, let

$$16)\quad v(x,t) = \int_{-A}^{A} \Gamma(x,\xi;\ t,0)\varphi(\xi)d\xi$$

$$\Gamma(x,\xi;\ t,0) \leqq k(x-\xi,t)[1 + Mt\ e^{Mt}]$$

where

$|c(x,t)| \leq M.$

$$v(x,t) \leq \int_{-\infty}^{\infty} k(x-\xi,t)[1 + Mt\, e^{Mt}]\varphi(\xi)d\xi.$$

For $|x| > 2A$, we have

$$17)\quad v(x,t) \leq \int_{-A}^{A} \frac{e^{-\frac{(|x|-A)^2}{4t}}}{\sqrt{4\pi t}} [1 + Mt\, e^{Mt}]\varphi(\xi)d\xi.$$

Now, since $|x| > 2A$, $\frac{|x|}{2} > A$, i.e. $-\frac{|x|}{2} < -A$,

or $\frac{|x|}{2} < |x| - A$,

we have

$$18)\quad v(x,t) \leq \frac{2A}{\sqrt{4\pi t}} e^{-\frac{x^2}{16t}}[1 + Me^{Mt}] \int_{-A}^{A} \varphi(\xi)d\xi$$

where $A > 1/2$.

Now $1 + Mte^{Mt} \leq 1 + MTe^{MT} \equiv B$. Then

$$19)\quad v(x,t) \leq \frac{2AB}{\sqrt{4\pi t}} e^{-\frac{x^2}{16t}} \int_{-A}^{A} \varphi(\xi)d\xi.$$

For fixed $x, F(t) = \frac{2AB}{\sqrt{4\pi t}} e^{-\frac{x^2}{16t}} \int_{-A}^{A} \varphi(\xi)d\xi$

is largest when $x^2 = 8t$. Thus

20) $v(x,t) \leqq \frac{2AB}{\sqrt{\frac{4\pi x^2}{8}}} e^{-\frac{8t}{16t}} \int_{-A}^{A} \varphi(\xi)d\xi.$

21) $v(x,t) \leqq \frac{2\sqrt{2}}{\sqrt{\pi e}} ABx^{-1} \int_{-A}^{A} \varphi(\xi)d\xi$

$$< ABx^{-1} \int_{-A}^{A} \varphi(\xi)d\xi.$$

In $t > 0$, $|x| \geqq 2AB + AB\varepsilon^{-1} \int_{-A}^{A} \varphi(\xi)d\xi = M(A,\varepsilon)$,

22) $0 \leqq v(x,t) < \dfrac{AB\int_{-A}^{A} \varphi(\xi)d\xi}{2AB + AB\varepsilon^{-1} \int_{-A}^{A} \varphi(\xi)d\xi}$

$$= \frac{\varepsilon\int_{-A}^{A} \varphi(\xi)d\xi}{2\varepsilon + \int_{-A}^{A} \varphi(\xi)d\xi} < \varepsilon.$$

Take any point (x_0,t_0) in $0 \leqq t \leqq T$. Let R be the rectangle with corners

$\{\pm M(A,\varepsilon),0\}$, $\{\pm M(A,\varepsilon),t_0\}$.

The function $u - v + \varepsilon$ is positive along the lower and vertical sides of the rectangle. It is positive along the vertical sides since there $u \geqq 0$ and $0 \leqq v \leqq \varepsilon$, and positive along the lower side, since there $u(x,0) = \varphi(x)$, while $v(x,0) = \varphi(x) -$ (i.e. some quantity less than $\varphi(x)$). Thus we have $u(x,t) - v(x,t) + \varepsilon \geqq 0$ along the lower and vertical sides of the rectangle R. Hence, by Nirenberg's strong maximum principle (see L. Nirenberg [7]), together with the following known result: namely, if $Lu = 0$ in $a < x < b$, $0 < t \leqq c$, and $\liminf u(x,t) \geqq 0$ as $(x,t)\to(x_0,t_0)$ when $x_0 = a$ or b, $0 \leqq t_0 \leqq c$, or $t_0 = 0$, $a \leqq x_0 \leqq b$, then $u(x,t) \geqq 0$, $a < x < b$, $0 < t \leqq c$, we have

23) $U(x_0;t_0) \geqq v(x_0,t_0) - \varepsilon$.

Since ε was arbitrary

24) $u(x_0,t_0) \geqq v(x_0,t_0)$.

A was also arbitrary, which proves (14).

We now derive an estimate of the form of theorem 3 for u.

Lemma 5. If $Lu = 0$ in $0 \leqq t < T$ where $-M \leqq c(x,t) \leqq 0$, $u(x,t) \geqq 0$, and $u(x,0) = 0$ in $-\infty < x < \infty$, then $u(x,t)$ vanishes identically in the strip $0 \leqq t < T$.

Proof: Set $w(x,t) = \int_0^t u(x,y)dy$. Let

25) $$Du = \frac{\partial^2 u}{\partial x^2} + c(x,t)u = \frac{\partial u}{\partial t}$$

26) $$Dw(x,t) = \int_0^t Du(x,y)dy = \int_0^t \frac{\partial}{\partial y} u(x,y)dy = u(x,t)$$

27) $$\frac{\partial}{\partial t} w(x,t) = u(x,t) \geqq 0.$$

Consequently, $Lw = 0$ in $0 \leqq t < T$. It satisfies all the conditions of our lemma, but has the additional properties of being convex in x and non-decreasing in t. A sufficient condition for the convexity of w is

$$\frac{\partial^2 w}{\partial x^2} \geqq 0$$

$$\frac{\partial^2 w}{\partial x^2} + c(x,t)w = \frac{\partial w}{\partial t}$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{w}{t} - c(x,t)w \geqq 0,$$

since $c(x,t)$ is negative and w is positive.

Moreover, if $w(x,t)$ vanishes identically, the same is true of $u(x,t)$. Hence, without loss of generality, we may assume that $u(x,t)$ is convex in x and non-decreasing in t.

Let δ be an arbitrary positive number less than T, and set $x = 0$, $t = t_0 < T - \delta$ in (15). Then

28) $$\int_{-\infty}^{\infty} \Gamma(0,\xi;\ t_0,0)u(\xi,\delta) < \infty,\quad 0 < \delta < T.$$

Since $u(x,t)$ is non-negative and non-decreasing in t

29) $$f(x) \equiv \max_{0 \leqq t \leqq \delta} u(x,t) = u(x,\delta).$$

By the convexity of $f(x)$ we have since

30) $$f(x) \leqq \frac{1}{2x} \int_0^{2x} f(y)dy, \text{ that}$$

31) $$2xf(x)e^{-x^2/t_0} \leqq e^{-x^2/t_0} \int_0^{2x} f(y)dy$$

$$\leqq \alpha \int_0^{2x} \Gamma(0,y;\ t_0,0)f(y)dy \leqq \int_{-\infty}^{\infty} \alpha\, \Gamma(0,y;\ t_0,0)f(y)dy$$

$$= \text{const} < \infty,$$

where the last integral converges by inequality (28).

Hence

32) $f(x) \leqq \frac{\text{const}}{2x} e^{x^2/t_0}$, or

33) $f(x) = O(e^{\beta x^2})$, where β is a constant.

Similarly the relation (32) holds for $x \to -\infty$. This gives the desired estimate. All hypotheses of theorem 4 are satisfied in the strip $0 \leqq t \leqq \delta$ so that $u(x,t)$ vanishes there. Since δ was arbitrary our lemma is proven.

Now, set

33a) $\tilde{u}(x,t) = \int_{-\infty}^{\infty} \Gamma(x,\xi;\ t,0)\varphi(\xi)d\xi.$

We must show that $u \equiv \tilde{u}$.

But by theorem 3,

34) $\lim\limits_{\substack{x \to x_0 \\ t \to t_0}} \tilde{u}(x,t) = u(x_0,0)$

and since u is continuous

35) $\lim\limits_{\substack{x \to x_0 \\ t \to t_0}} u(x,t) = u(x_0,0).$

Therefore, by lemma 5,

36) $u \equiv \tilde{u}$.

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APPENDIX

APPENDIX

We shall now discuss briefly a corresponding problem for the half plane, namely

37) $Lu = 0$

$u(x,0) = \varphi(x)$

$u(0,t) = 0.$

By proceeding formally in the same manner as is done in the case of the heat equation (see G. Hellwig [6]), we arrive at

$$38) \quad u(x,t) = \int_0^\infty [\Gamma(x,\xi;\ t,0)-\Gamma(x,-\xi;\ t,0)]\varphi(\xi)d\xi$$

as a solution to (37). We now proceed in much the same way as we did for the previous problem (5). We prove analogous theorems and lemmas: however, in this case we would also have to show that the representation in (38) yields $u(0,t)= 0$. Thus we may formulate for (37) the following theorem analogous to theorem 5.

Theorem 6. If $Lu = 0$, and $u(x,t)$ is continuous for $0 \leqq t \leqq T$ and $u(x,t) \geqq 0$, where $-M \leqq c(x,t) \leqq 0$ and where c satisfies a uniform Hölder condition of order γ, then

$$u(x,t) = \int_{-\infty}^\infty [\Gamma(x,\xi;\ t,0)-\Gamma(x,-\xi;\ t,0)]\varphi(\xi)d\xi$$

where $u(\xi,0) = \varphi(\xi)$ and $u(0,t) = 0$. Thus $u(x,t)$ is uniquely determined by $\varphi(\xi)$.