Generalized Quadratic Revenue Functions*

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Abstract: In this paper we focus on the specification of revenue functions in their dual price space. We consider two distance functions—the Shephard output distance function and the directional output distance function—and define both in price space. The former is multiplicative in nature and satisfies homogeneity, whereas the latter is additive and satisfies the translation property. Functional equation methods yield the translog specification in the case of the Shephard distance function and a quadratic specification in the case of the directional distance function. Monte Carlo evidence suggests that the quadratic specification outperforms the translog in large samples and in true models with plenty of curvature.

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1. Introduction

With the introduction of the Shortage Function (Luenberger, 1992, 1995) or Directional Distance Functions (Chambers, Chung, and Färe, 1996) into economics, we have a new tool at our disposal for characterizing technology (or consumer preferences). These distance functions satisfy translation, a property that follows from their definition, and which corresponds to the more familiar homogeneity conditions that are characteristic of Shephard's distance functions. Both types of functions accommodate multiple inputs and outputs, which has proven useful in the performance measurement literature.

An appealing feature of the distance functions is the fact that they have well-known economic dual representations. For example, the revenue function is a Shephard (1970) type output distance function in price space. Here we derive the companion directional output distance function in price space. We then propose to compare these two price space distance functions in terms of their ability to represent technology in price space. Our approach is to parameterize these distance functions within the family of generalized quadratic functions and undertake a Monte Carlo experiment to assess their relative ability to describe the price space technology. Since both distance functions fully characterize the price space technology, we have two alternatives for its representation. The Monte Carlo experiment will provide guidance as to which distance function performs better empirically.

In a recent paper Färe, Martins-Filho, and Vardanyan (2010) use a similar research design to compare the performance of Shephard and Directional Distance Functions in output quantity space in a production context. Based on their Monte Carlo experiment they conclude that the directional distance function does a better job of modeling the technology within the family of generalized quadratic functions in quantity space. Färe et al. (2009) study these functions in the consumer theory context and come to a similar conclusion.

We exploit the translation and homogeneity properties to help us choose appropriate functional forms for parameterizing our distance functions. We make use of functional equation solutions by Färe and Sung (1986) and Färe and Lundberg (2006) to identify the functional forms that satisfy the aforementioned properties as well as flexibility and linearity in parameters, i.e. they satisfy properties from economic theory as well as providing practical empirical properties. In contrast, in a recent study Feng and Serletis (2008) claim "… there is no *a priori* view as to which flexible functional forms are appropriate…." Based on the results of Färe and Sung (1986) and Färe and Lundberg (2006) we use a Monte Carlo study to compare which functional form is appropriate for estimating technologies in price space.

2. Parameterizing Generalized Quadratic Functions

In this section we discuss the generalized quadratic function—a flexible functional form—and show how it may be parameterized using homogeneity and translation properties. In other words, the generalized quadratic function belongs to a class of functions that nest translation and homogeneity properties. Recall that the revenue function is *homogeneous* in output prices and that it is an output distance function in price space. The 'new' revenue function introduced in this paper as a directional output distance function in price space is shown below to satisfy the *translation* property. Homogeneity and translation are the properties we use to help us parameterize these functions.

Let $F:\Re^I \rightarrow \Re$, $h:\Re \rightarrow \Re$ and $\varsigma:\Re \rightarrow \Re$ with an inverse ς^{-1}. If a_i, a_{ij} are real constants and $q_i \in \Re_+$ then

$$\varsigma^{-1}(F(q)) = a_0 + \sum_{i=1}^{I} a_i h(q_i) + \sum_{i=1}^{I}\sum_{j=1}^{I} a_{ij} h(q_i) h(q_j), \tag{1}$$
$$a_i, a_{ij} \in \Re$$

is called a generalized quadratic function (Chambers, 1988), a transformed quadratic function (Diewert, 2002) or is said to have a second-order Taylor's series approximation interpretation (Färe and Sung, 1986). If $a_i = 0, \; i = 1,...,I$ and $a_{ij} \neq 0, \; i = 1,...,I$ then it collapses to a so-called generalized quasi-quadratic function (Färe and Sung, 1986).

We say that $F(q)$ is homogeneous of degree +1 if

$$F(\lambda q) = \lambda F(q), \quad \lambda > 0, \tag{2}$$

and it satisfies the translation property if

$$F(q + \alpha g) = F(q) + \alpha, \quad \alpha \in \Re, \tag{3}$$

where $g = (g_1,..., g_I) \in \Re^I, g \neq 0$ is the directional, or mapping, vector.[1]

[1] In efficiency analysis this is the direction in which efficiency and productivity are measured (Chambers, Chung, and Färe, 1996).

Note that the generalized quadratic function (1) is linear in the parameters a_i and a_{ij} and that it is quadratic in $h(\cdot)$. The first property is desirable from an econometric point of view and the second from an economic point of view.

The interactions between (1) and (2) or between (1) and (3) yield functional equations. What we seek are the solutions to these functional equations, which will provide the 'functional form' that globally satisfies the conditions (1) and (2) or (1) and (3). As it turns out, there are only two solutions for each pair of conditions, which provide the basis for our choice of parameterization. Beginning with (1) and (2), or our generalized form in combination with homogeneity, one can obtain the following solutions (see Färe and Sung, 1986):

$$F(q) = a_0 + \sum_{i=1}^{I} a_i \ln(q_i) + \sum_{i=1}^{I}\sum_{j=1}^{I} a_{ij} \ln(q_i)\ln(q_j), \tag{4}$$

also known as the translog function (Christensen et al., 1971), and

$$F(q) = \left(a_0 + \sum_{i=1}^{I}\sum_{j=1}^{I} a_{ij} q_i^{r/2} q_j^{r/2}\right)^{1/r}, \tag{5}$$

which is the quadratic mean of order *r* function (Denny, 1974, Diewert, 1976).[2]

[2] See also Diewert (1971), who introduced the generalized Leontief function

$$F(q) = \sum_{i=1}^{I} a_i q_i + \sum_{i=1}^{I}\sum_{j=1}^{I} a_{ij} q_i^{1/2} q_j^{1/2}.$$

The functional equations (1) and (3), or the combination of the generalized quadratic function and the translation property, also yield two solutions (Färe and Lundberg, 2006). Assuming $g = (1,...,1)$, we have:

$$F(q) = a_0 + \sum_{i=1}^{I} a_i q_i + \sum_{i=1}^{I}\sum_{j=1}^{I} a_{ij} q_i q_j , \tag{6}$$

the quadratic function,[3] and

$$F(q) = \frac{1}{2\lambda} \ln \sum_{i=1}^{I}\sum_{j=1}^{I} a_{ij} \exp(\lambda q_i)\exp(\lambda q_j), \quad \lambda \neq 0.^{4} \tag{7}$$

Note that several additional constraints must be imposed on the coefficients of functions given by (4), (5), and (6) in order to satisfy homogeneity or translation. For example, homogeneity of the translog function in (4) and translation of the quadratic function in (6)—the two functional forms that will play key roles in our study—is established via $\sum_i a_i = 1$ and $\sum_i a_{ij} = \sum_j a_{ij} = 0$. Homogeneity of the quadratic mean of order *r* function in (5) requires

[3] See also Diewert and Wales (1988), who introduced the normalized quadratic function, given by $F(q) = \sum_{i=1}^{I} a_i q_i + \dfrac{\sum_{i=1}^{I}\sum_{j=1}^{I} a_{ij} q_i q_j}{2\sum_{i=1}^{I} b_i q_i}$.

[4] In a private conversation Professor Diewert names it “Quadratic exponential mean of order *s*,” here λ. Kolm (1976) and Diewert and Wales (1993) called it exponential mean or order *s*, here λ.

$a_0 = 0$. Finally, translation of the function given by (7) is satisfied without imposing any additional constraints on its coefficients.

3. Generalized Quadratic Revenue Functions

Let $x \in \Re_+^N$ denote inputs and $y \in \Re_+^M$ outputs; we model technology here by its output sets

$$P(x) = \{y : x \quad can \quad produce \quad y\}, \quad x \in \Re_+^N. \tag{8}$$

We assume that the output sets satisfy the usual axioms including free disposability of inputs and outputs, non-emptiness, and compactness for $x \in \Re_+^N$.[5]

Let $p \in \Re_+^M$ be an output price vector, with the corresponding revenue function defined as

$$R(x, p) = \max\{py : y \in P(x)\}, \quad x \in \Re_+^N. \tag{9}$$

This function is homogeneous of degree +1 in output prices:

$$R(x, \lambda p) = \lambda R(x, p), \quad \lambda > 0. \tag{10}$$

If R is a given revenue value then the associated output set in price space is

$$\wp(x, R) = \{p : R(x, p) \le R\}. \tag{11}$$

[5] See Färe and Primont (1995) for details.

These price output sets are closed, convex, and monotonic (see Shephard, 1970). We note that the revenue function is an output distance function in price space, i.e.:

$$
\begin{aligned}
D(x,p,R) \quad &= \inf\{\lambda : (p/\lambda) \in \wp(x,R)\} \\
&= \inf\{\lambda : R(x,p/\lambda) \leq R\} \\
&= \inf\left\{\lambda : \frac{R(x,p)}{R} \leq \lambda\right\} \\
&= \frac{R(x,p)}{R}.
\end{aligned} \tag{12}
$$

The second equality follows from the definition of the output set in price space and the third from the homogeneity of the revenue function in output prices. Next, let $g = (g_1,\ldots,g_M) \in \Re^M_+$, $g \neq 0$, be a directional vector, then the directional revenue function is defined as

$$
\begin{aligned}
\Delta(x,p,R;g) \quad &= \sup\{\beta : (p+\beta g) \in \wp(x,R)\} \\
&= \sup\{\beta : R(x,p+\beta g) \leq R\}.
\end{aligned} \tag{13}
$$

Note that if $p = g$ then

$$
\begin{aligned}
\Delta(x,p,R;g) \quad &= \sup\{\beta : R(x,p(1+\beta)) \leq R\} \\
&= -1 + \sup\{(1+\beta) : R(x,p)(1+\beta) \leq R\} \\
&= \frac{R}{R(x,p)} - 1 \\
&= \frac{1}{D(x,p,R)} - 1,
\end{aligned} \tag{14}
$$

which shows the relationship between the directional revenue function and the revenue function $R(x, p)$, or, equivalently, the price output distance function $D(x, p, R)$.

The price output distance function and the directional revenue function are illustrated in Figure 1. Both panels are of the same output price technology, $\wp(x, R)$, and evaluate the observed output price pair at *A*. The Shephard price output distance function, which takes values in the interval $(0,1]$, projects *A* to the frontier of technology along a ray from the origin, i.e., for observation *A* $D(x, p, R) = 0A/0A'$. The directional revenue function for observation *A* is illustrated in the bottom panel; here the problem is to maximize $(p_1, p_2) + \beta g$ with respect to *β*, where *g* is the directional vector which we add to *A*. We then scale along the segment *A*, $g + A$ until we reach the frontier at $(p_1, p_2) + \Delta(x, p, R; g)g$. If we assumed $g = (p_1, p_2)$ then the directional vector would lie on the ray from the origin, and the directional revenue function would be equal to $\frac{1}{D(x, p, R)} - 1$.

From its definition, it follows that $\Delta(x, p, R; g)$ satisfies the translation property, i.e.[6]

$$\Delta(x, p + \alpha g, R; g) = \Delta(x, p, R; g) - \alpha. \tag{15}$$

The following lemma establishes the relationship between the two revenue functions $\Delta(x, p, R; g)$ and $R(x, p)$. The proof is given in the appendix.

LEMMA: $\Delta(x, p, R; g) = 0 \Leftrightarrow R(x, p) = R.$

[6] For the case in which $g = (1,...,1)$ (15) corresponds to (3).

Under the appropriate disposability assumptions, such as strong disposability of prices, both revenue functions completely characterize the price space technology $\wp(x, R)$. Therefore, either one can be used to model this technology, although the directional revenue function has the distinct advantage of accommodating zeros as arguments.

Using the above lemma we may write the directional revenue function as

$$\Delta(x, p, R(x, p); g) = 0. \tag{16}$$

If the revenue functions are differentiable we may derive the following marginal conditions from (16):

$$\partial R/\partial p_m = -\frac{\partial \Delta/\partial p_m}{\partial \Delta/\partial R}, m = 1,..., M \tag{17}$$

and

$$\partial R/\partial x_n = -\frac{\partial \Delta/\partial x_n}{\partial \Delta/\partial R}, n = 1,..., N, \tag{18}$$

where (17) yields the supply function for outputs and (18) gives us the shadow prices of inputs. Note that the left hand side is based on the conventional revenue function whereas the right hand side depends on the directional revenue function. Hence, one may estimate $\Delta(x, p, R; g)$ and then derive the desirable properties of $R(x, p)$.

To illustrate this derivation we provide a simple example of (16). Let technology be a simple production function

$$F(x) = \max\{y : y \in P(x)\}, y \in \Re_+ \tag{19}$$

with

$$y = \sqrt{x}. \tag{20}$$

The corresponding revenue function is

$$R(x, p) = p\sqrt{x}. \tag{21}$$

The directional revenue function corresponding to $g = 1$ is

$$\begin{aligned} \Delta(x, p, R; 1) \quad &= \max\{\beta : (p + \beta)\sqrt{x} \leq R\} \\ &= \max\{\beta : p\sqrt{x} + \beta\sqrt{x} \leq R\} \\ &= \max\left\{\beta : \beta \leq \frac{R}{\sqrt{x}} - p\right\} \\ &= \frac{R}{\sqrt{x}} - p. \end{aligned} \tag{22}$$

The supply function associated with (21) equals

$$\partial R / \partial p = \sqrt{x}. \tag{23}$$

From (22) we have

$$\partial \Delta / \partial p = -1 \tag{24}$$

and

$$\partial\Delta / \partial R = 1 / \sqrt{x}\,, \tag{25}$$

so that

$$-\frac{\partial\Delta / \partial p}{\partial\Delta / \partial R} = \sqrt{x} = \partial R / \partial p\,. \tag{26}$$

In addition to the derivation of the supply function for outputs and the shadow prices of inputs, the directional revenue function can be used to answer a number of other relevant questions about the price space technology. For example, it can be used in the context of the Luenberger indicator (Chambers, 1996, 2002), defined with respect to directional distance functions, in order to measure changes in productivity in price space.

In our Monte Carlo experiment, the conventional revenue function is parameterized using the translog functional form due to its homogeneity. We parameterize the directional revenue function as a quadratic, consistent with the translation property of the directional revenue function. We choose translog and quadratic over the other possible solutions because these functional forms have both first and second order terms.

4. The Monte Carlo Experiments

To assess the approximation properties of the quadratic and translog function we adhere to the tradition of Wales (1977), Guilkey and Lovell (1980) and Guilkey et al. (1983), who used Monte Carlo experiments to study the performance of several parameterizations of the cost function, notably translog.[7] We follow the setup of the experiment outlined in Färe, Martins-

[7] The other approach, introduced by Caves and Christensen (1980) and extended by Barnett and Lee (1985), is more analytical in nature. It is based on the comparisons of regions of the true

Filho, and Vardanyan (2010), and our Monte Carlo experiments focus on two classes of the true price space technologies. We assume three so-called polynomial-of-order-four technologies (P1, P2, P3), which give us our 'translation' quadratic technology, and three translog-of-order-four technologies (L1, L2, L3), which satisfy homogeneity. We assume that two inputs are used to generate two output prices and use the normalization $R=1$ for all simulated observations in our samples. This normalization will aid in the visual assessment of the quality of approximation, as the price output set is specified for a given level of revenue and input utilization.

Beginning with the polynomial-of-order-four technologies (P), we have

$$\wp^{P}(x,1)=\left\{\left(p_1,p_2\right):p_2=f^{P}(x,p_1)\right\}, \tag{27}$$

where $f^{P}(x,p_1)=\beta_0^P+\beta_1^P p_1+\beta_2^P p_1^2+\beta_3^P p_1^3+\beta_4^P p_1^4+\sqrt{x_1}x_2^{0.1}$ defines the frontier of the price output set[8], the parameter vector $\beta^P=\left(\beta_0^P,\ldots,\beta_4^P\right)$ models the degree of its concavity, and $x\in\Re_+^2$. The three assumed scenarios cover a wide range of possibilities and at the same time allow for a relatively simple interpretation of the simulation results. The parameters are chosen in the following way:

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technology that can be approximated by various functional forms without violations of regularity conditions.

[8] During the initial stages of our research we experimented with a number of additional cases involving other shapes of the true price output set boundaries. The outcomes of these experiments are very similar to the results that we describe in this section.

Polynomial-of-Order-Four Technologies			
	Model P1	Model P2	Model P3
β_0^P	10.70	10.10	9.60
β_1^P	-0.91	-0.72	-0.54
β_2^P	0.50×10^{-5}	0.50×10^{-4}	0.10×10^{-2}
β_3^P	0.10×10^{-4}	0.10×10^{-3}	0.10×10^{-2}
β_4^P	-0.45×10^{-3}	-0.12×10^{-2}	-0.24×10^{-2}

Note that this setup can be extended to a more general case involving more than two production inputs and/or a varying revenue. However, given the goals of the experiment, this generalization is not necessary, since the above setup already includes such possibilities through our choice of the parameter vector β^P. Panel (A) of Figure 2 illustrates the plots of the price output set frontiers for the valid range of the first output price and $x_1 = x_2 = 1$. Model P1 has the 'flattest' price output frontier, and Model P3 has the most curvature.

The prices p_1 are generated by drawing the samples of various sizes (K) from a gamma distribution with the density given by $f(p_1) = p_1^{\lambda-1}e^{-p_1/\theta}\left(\Gamma(\lambda)\theta^{\lambda}\right)^{-1}$, where $\Gamma(\cdot)$ is the gamma function, with $(\lambda,\theta)\in\Re_+^2$. Simulations are performed using sample sizes of 50, 100, and 500 observations.

Our class of polynomial technologies is further divided into two subclasses, type-A and type-B models, which differ by the values of the parameter vector (λ,θ) that we assume for the experiment. Specifically, type-A specifications have $(\lambda,\theta) = (5,\ \ 0.5)$, whereas type-B models assume $(\lambda,\theta) = (18,\ \ 0.25)$.[9] In both subclasses the production inputs are randomly drawn from

[9] The type-A parameters yield data with relatively low values of p_1 and relatively high values of p_2, whereas type-B parameters yield relatively more "balanced" prices.

the standard uniform distribution. The prices p_2 for the polynomial technologies are then generated as $p_2 = f^P(x, p_1) - \nu + \varepsilon$, where $\nu \sim |N(0,1)|$ captures the price-space counterpart of 'technical inefficiency' and $\varepsilon \sim N(0,1)$ introduces specification errors.

Turning next to the specification of the translog price technologies (L), we have

$$\wp^L(x,1) = \{(p_1, p_2) : \ln(p_2) = f^L(x, p_1)\}, \tag{28}$$

where $f^L(x, p_1) = \beta_0^L + \beta_1^L \ln(p_1) + \beta_2^L [\ln(p_1)]^2 + \beta_3^L [\ln(p_1)]^3 + \beta_4^L [\ln(p_1)]^4 + \sqrt{x_1}\, x_2^{0.1}$, and the parameter vector $\beta^L = (\beta_0^L, \ldots, \beta_4^L)$ is chosen in the following way:[10]

Translog-of-Order-Four Technologies			
	Model L1	Model L2	Model L3
β_0^L	2.000	1.845	1.690
β_1^L	-3.500	-3.400	-3.300
β_2^L	3.900	4.000	4.100
β_3^L	-1.500	-1.475	-1.415
β_4^L	-0.140	-0.220	-0.330

As in the case of the polynomial-of-order-four technologies, our goal will be to assess how well the quadratic and translog functional forms approximate this class of true technologies.

[10] Both β^L and β^P are assumed to be the same as in Färe, Martins-Filho, and Vardanyan (2010). Note further that in order to facilitate the comparison between the two classes of true models we assumed that prices are given in logs but the inputs are expressed in quantities in the case of the translog-of-order-four technologies.

Panel (B) of Figure 2 illustrates the plots of the corresponding boundaries for the translog case. We draw the samples of the same three sizes as before from the uniform distribution as $\ln(p_1) \sim U(0.7,\ 1.4)$ to ensure that all of the true price output set frontiers have non-decreasing price-space counterparts of the marginal rate of transformation at each value of p_1. As in the case of the polynomial technologies, the three choices allow us to keep the experiment both reasonably general as well as easily interpretable. Finally, the price-space counterpart of 'technical inefficiency' and the conventional disturbance term are introduced in a similar way as before, i.e. $\ln(p_2) = \ln\left(\exp\{f^L(x, p_1)\} - \nu + \varepsilon\right)$.

To obtain the estimates of these true frontiers we rely on the stochastic frontier algorithm of Aigner et al. (1977) and Meeusen and van den Broeck (1977) and estimate the parameters of the translog price output distance function and the quadratic directional revenue function using maximum-likelihood. Consistent with a number of previous empirical studies of distance functions, we follow the practice established in the literature and rely on the homogeneity of the former and the translation of the latter in order to obtain suitable specifications that can be estimated econometrically.[11] For example, expressed in logs, homogeneity of the price output distance function implies the following:

$$\ln\left(D(x, \lambda p_1, \lambda p_2, 1)\right) = \ln(\lambda) + \ln\left(D(x, p_1, p_2, 1)\right). \tag{29}$$

[11] See, for example, Grosskopf et al. (1997) and Atkinson et al. (2003a, 2003b), who demonstrate how to obtain estimable specifications by incorporating the homogeneity property. Färe et al. (2005) and Koutsomanoli-Filippaki et al. (2009) are among the studies that show how the translation property can instead be used for the same purpose.

Since the price output distance function takes values in the interval $(0,1]$, we start by assuming that $\exp\{\nu\} = [D(x, p_1, p_2, 1)]^{-1}$ and then introduce the conventional error via $\exp\{\nu\} = [D(x, p_1, p_2, 1)]^{-1}\exp\{\varepsilon\}$. Taking the logs of both sides of this expression and rearranging yields

$$-\ln(D(x, p_1, p_2, 1)) = \nu - \varepsilon \,. \tag{30}$$

Finally, inserting (30) into (29), assuming $\lambda = 1/p_2$, rearranging once again, and suppressing the normalized revenue for notational convenience produces

$$-\ln(1/p_2) = -\ln(D(x, p_1/p_2)) - \nu + \varepsilon \,. \tag{31}$$

We parameterize the function $\ln(D(x, p_1/p_2))$ using the translog functional form and then apply the stochastic frontier algorithm of Aigner et al. (1977) in a straightforward fashion in order to estimate specification (31) using maximum-likelihood.[12] The estimated coefficients corresponding to $\ln(D(x, p_1/p_2))$ can then be used to identify all of the parameter estimates of the associated translog price output distance function, given by

[12] The log-likelihood function corresponding to the composed error model in (31) has been used in a large number of studies and is therefore not reported here. It can be found in the influential paper of Aigner et al., as well as in a number of other manuscripts and books.

$$
\begin{aligned}
\ln\left(D\left(x, p_1, p_2\right)\right) = \gamma_0 + \gamma_1 \ln(p_1) + \gamma_2 \ln(p_2) + \frac{\gamma_{11}}{2}\left(\ln(p_1)\right)^2 + \frac{\gamma_{22}}{2}\left(\ln(p_2)\right)^2 + \gamma_{12} \ln(p_1)\ln(p_2) \\
+\gamma_3 \ln(x_1) + \gamma_4 \ln(x_2) + \frac{\gamma_{33}}{2}\left(\ln(x_1)\right)^2 + \frac{\gamma_{44}}{2}\left(\ln(x_2)\right)^2 + \gamma_{34} \ln(x_1)\ln(x_2) \\
+\gamma_{13} \ln(p_1)\ln(x_1) + \gamma_{14} \ln(p_1)\ln(x_2) + \gamma_{23} \ln(p_2)\ln(x_1) + \gamma_{24} \ln(p_2)\ln(x_2),
\end{aligned}
\tag{32}
$$

since $\ln\left(D\left(x, p_1, p_2\right)\right) = -\nu$.

Turning next our attention to the directional revenue function, which takes values in the interval $\left[0; +\infty\right)$, we can assume that $\nu = \Delta\left(x, p_1, p_2, 1; g\right)$. Adding the conventional error to the right-hand side of this expression and rearranging yields

$$\Delta\left(x, p_1, p_2, 1; g\right) = \nu - \varepsilon . \tag{33}$$

As before, plugging (33) into (15), taking $\alpha = -p_2$, suppressing the normalized revenue, and rearranging will produce the following estimable econometric specification when $g = (1,1)$:[13]

$$p_2 = \Delta\left(x, p_1 - p_2, 0\right) - \nu + \varepsilon . \tag{34}$$

After the function $\Delta\left(x, p_1 - p_2, 0\right)$ has been parameterized using the quadratic functional form, equation (34) can be estimated using the same maximum-likelihood algorithm as before.[14] By

[13] Other econometric specifications can be obtained by assuming different directional vectors. For example, when $g = (1,3)$ the corresponding econometric specification is given by $p_2/3 = \Delta\left(x, \left(p_1 - p_2/3\right), 0\right) - \nu + \varepsilon$.

relying on the assumption $\nu = \Delta(x, p_1, p_2; g)$, we can proceed to recover the parameter estimates associated with the quadratic directional revenue function itself:

$$\Delta(x, p_1, p_2) = \delta_0 + \delta_1 p_1 + \delta_2 p_2 + \frac{\delta_{11}}{2} p_1^2 + \frac{\delta_{22}}{2} p_2^2 + \delta_{12} p_1 p_2 + \delta_3 x_1 + \delta_4 x_2 \\ + \frac{\delta_{33}}{2} x_1^2 + \frac{\delta_{44}}{2} x_2^2 + \delta_{34} x_1 x_2 + \delta_{13} p_1 x_1 + \delta_{14} p_1 x_2 + \delta_{23} p_2 x_1 + \delta_{24} p_2 x_2, \tag{35}$$

Finally, we perform 500 replications for each specification and consider three directional vectors: $g = (3,1)$, $g = (1,1)$, and $g = (1,3)$. Each of the vectors corresponds to a separate estimate of the price output set frontier, and their comparison should shed light on the possible sensitivity of results to the choice of the mapping regime in quadratic models.

In the next stage of the experiment we rely on the maximum-likelihood estimates of the parameters associated with (32) and (35) to obtain estimated price-space frontiers for each of our nine true models. We then compare the quality of the approximation provided by the translog and the quadratic parameterizations. We assume a fixed level of inputs for all observations, i.e. $x_{nk} = \bar{x}_n$, then assume price-space technical efficiency and no specification error for every observation in the sample, i.e. $\Delta^k(\bar{x}, p, 1; g) = 0$ and $D^k(\bar{x}, p, 1) = 1$ for all $k = 1,...,K$. We then use the actual quantities p_1 to solve for the corresponding optimal values of the second price, or $p_{2k}^*(\bar{x}, \hat{\gamma})$ and $p_{2k}^*(\bar{x}, \hat{\delta})$. These price pairs place every observation on the estimated boundary of the price output set producing the empirical analogue of its plot.

The following three benchmarks are then used to assess the quality of parametric approximations:

[14] Note that the translog function cannot be used in this case due to the zero argument.

(1) The average Euclidean distance between the true and simulated prices of the second output.

(2) The average discrepancy between the true and estimated price-space counterparts of the relative shadow prices.

(3) The mean Euclidean distance between the true and estimated price-space measures of the frontier curvature. This measure can be interpreted as the price-space counterpart of the Morishima elasticity of substitution [Morishima (1967)].

The first benchmark is obtained using the true and the simulated prices of the second output and is given by

$$\overline{\Theta}(\hat{\gamma}) = K^{-1}\left(\sum_{k=1}^{K}\left[d_k + p_{2k}^{*}(\overline{x},\hat{\gamma})\right]^2\right)^{1/2} \tag{36}$$

and

$$\overline{\Theta}(\hat{\delta}) = K^{-1}\left(\sum_{k=1}^{K}\left[d_k + p_{2k}^{*}(\overline{x},\hat{\delta})\right]^2\right)^{1/2}, \tag{37}$$

where $d_k = -\exp\left\{f^{L}(\overline{x}, p_{1k})\right\}$ or $d_k = -f^{Q}(\overline{x}, p_{1k})$ corresponds to the second price in 'true' translog or 'true' polynomial models, respectively.

The second benchmark can be interpreted as the average discrepancy between the true and estimated price-space counterparts of the marginal rate of transformation. These quantities are computed for every observation and evaluated at the frontiers of the estimated price output sets. From duality theory, the relative shadow price can be defined as [Färe and Primont (1995); Färe and Grosskopf (2004)]

$$\psi = \frac{\partial D(x,p,1)/\partial p_1}{\partial D(x,p,1)/\partial p_2} = \frac{\partial \Delta(x,p,1;g)/\partial p_1}{\partial \Delta(x,p,1;g)/\partial p_2}. \tag{38}$$

Hence, the average Euclidean distance between the true and estimated price-space representations of relative shadow prices evaluated at corresponding frontier points is equal to

$$\overline{\Omega}(\hat{\gamma}) = K^{-1}\sum_{k=1}^{K}\left(\left[\rho_k + \frac{\partial \ln D\left(\overline{x}, p_{1k}, p_{2k}^*(\overline{x},\hat{\gamma}),1\right)/\partial \ln(p_1)}{\partial \ln D\left(\overline{x}, p_{1k}, p_{2k}^*(\overline{x},\hat{\gamma}),1\right)/\partial \ln(p_2^*)}\frac{\exp\{p_2^*(\overline{x},\hat{\gamma})\}}{p_{1k}}\right]^2\right)^{1/2} \tag{39}$$

and

$$\overline{\Omega}(\hat{\delta}) = K^{-1}\sum_{k=1}^{K}\left(\left[\rho_k + \frac{\partial \Delta\left(\overline{x}, p_{1k}, p_{2k}^*\left(\overline{x},\hat{\delta}\right),1;g\right)/\partial p_1}{\partial \Delta\left(\overline{x}, p_{1k}, p_{2k}^*\left(\overline{x},\hat{\delta}\right),1;g\right)/\partial p_2^*}\right]^2\right)^{1/2}, \tag{40}$$

where ρ_k is the true shadow price for observation *k*. Note that $\rho_k = -\dfrac{\partial f^{Q}(\overline{x}, p_{1k})}{\partial p_1}$ in the case of polynomial technologies and $\rho_k = -\dfrac{\partial f^{L}(\overline{x}, p_{1k})}{\partial \ln(p_1)}\dfrac{\exp\{f^{L}(\overline{x}, p_{1k})\}}{p_{1k}}$ for translog technologies.

Finally, our third benchmark assesses the relative error in the approximation of the price output set curvature. It is defined as $\partial \ln(\psi)/\partial \ln(p_2/p_1)$ and we have

$$\begin{aligned} e_k(\hat{\gamma}) &= 1 - \frac{\partial^2 \ln D\left(\overline{x}, p_{1k}, p_{2k}^*(\overline{x},\hat{\gamma}),1\right)/\partial(\ln(p_1))^2}{\partial \ln D\left(\overline{x}, p_{1k}, p_{2k}^*(\overline{x},\hat{\gamma}),1\right)/\partial \ln(p_1)} + \frac{\partial^2 \ln D\left(\overline{x}, p_{1k}, p_{2k}^*(\overline{x},\hat{\gamma}),1\right)/\partial \ln(p_1)\partial \ln(p_2^*)}{\partial \ln D\left(\overline{x}, p_{1k}, p_{2k}^*(\overline{x},\hat{\gamma}),1\right)/\partial \ln(p_2^*)} \\ &= 1 - \frac{\hat{\gamma}_{11}}{\hat{\gamma}_1 + \hat{\gamma}_{11}\ln(p_{1k}) + \hat{\gamma}_{12}\ln(p_{2k}^*) + \hat{\gamma}_{13}\ln(x_{1k}) + \hat{\gamma}_{14}\ln(x_{2k})} \\ &\quad + \frac{\hat{\gamma}_{12}}{\hat{\gamma}_2 + \hat{\gamma}_{22}\ln(p_{2k}^*) + \hat{\gamma}_{12}\ln(p_{1k}) + \hat{\gamma}_{23}\ln(x_{1k}) + \hat{\gamma}_{24}\ln(x_{2k})} \end{aligned} \tag{41}$$

and

$$e_k(\hat{\delta}) = p_1\left(\frac{\partial^2\Delta\left(\bar{x}, p_{1k}, p_{2k}^*\left(\bar{x},\hat{\delta}\right),1; g\right)/\partial p_1\partial p_2^*}{\partial\Delta\left(\bar{x}, p_{1k}, p_{2k}^*\left(\bar{x},\hat{\delta}\right),1; g\right)/\partial p_2^*} - \frac{\partial^2\Delta\left(\bar{x}, p_{1k}, p_{2k}^*\left(\bar{x},\hat{\delta}\right),1; g\right)/\partial {p_1}^2}{\partial\Delta\left(\bar{x}, p_{1k}, p_{2k}^*\left(\bar{x},\hat{\delta}\right),1; g\right)/\partial p_1}\right)$$

$$= p_1\left(\frac{\hat{\delta}_{12}}{\hat{\delta}_2 + \hat{\delta}_{22}p_{2k}^* + \hat{\delta}_{12}p_{1k} + \hat{\delta}_{23}x_{1k} + \hat{\delta}_{24}x_{2k}} - \frac{\hat{\delta}_{11}}{\hat{\delta}_1 + \hat{\delta}_{11}p_{1k} + \hat{\delta}_{12}p_{2k}^* + \hat{\delta}_{13}x_{1k} + \hat{\delta}_{14}x_{2k}}\right). \tag{42}$$

As in the case of the second benchmark, frontier prices were used to obtain these quantities. Therefore, the mean Euclidean distance is equal to

$$\overline{\mathrm{E}}(\cdot) = K^{-1}\sum_{k=1}^{K}\left(\left[e_k + e_k(\cdot)\right]^2\right)^{1/2}, \tag{43}$$

where e_k is the true price-space elasticity of substitution for observation k. Also, note that

$e_k = -\dfrac{\partial^2 f^L\left(\bar{x}, p_{1k}\right)/\partial\ln\left(p_1\right)^2}{\partial f^L\left(\bar{x}, p_{1k}\right)/\partial\ln\left(p_1\right)} + 1$ and $e_k = -p_{1k}\dfrac{\partial^2 f^Q\left(\bar{x}, p_{1k}\right)/\partial {p_1}^2}{\partial f^Q\left(\bar{x}, p_{1k}\right)/\partial p_1}$ for true translog and true polynomial technologies, respectively. Note that the elasticity of substitution must be negative, reflecting the concavity of the price output set frontier.

Finally, our three benchmarks can be combined into an average measure of discrepancy, which will allow us to easily interpret the simulation results and shed light on the relative quality of approximation achieved by each of the parameterizations.

5. Results

Tables 1 and 2 contain the summary of our simulation results. The average discrepancy reported in Table 1 was obtained by giving our three benchmarks equal weights. Three quadratic directional revenue functions, based on directional vectors $g = (3,1)$, $g = (1,1)$, and $g = (1,3)$, were estimated along with the translog price output distance function for each of the nine true price space technologies. Since no curvature constraints were imposed during estimation, some simulations produced convex frontier estimates, contradicting the assumption of the concavity of the price output set frontier.[15] The fraction of such unexpected estimates is reported in Table 2.

First, the approximation quality corresponding to both parameterizations seems to deteriorate with an increase in the curvature of the associated true frontiers—a result that holds in all types of true models and for samples of any size. For example, all of the mean benchmark discrepancies in column 6 of Table 1 (Model P3A) are greater than their respective counterparts in column 4 (Model P2A), which are in turn greater than the corresponding column 2 entries (Model P1A). Comparing values in columns 7, 5, and 3, as well as columns 10, 9, and 8 points to the same conclusion in the case of type-B polynomial (similar range for p_1 and p_2) and true translog technologies, respectively. These results are hardly surprising, since our true models are

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[15] Our decision not to impose any curvature constraints was motivated by several reasons. Gagné and Ouellette (1998) found disappointing deteriorations in the approximation properties of two of the three functional forms they considered, namely symmetric McFadden and symmetric generalized Barnett, caused by the imposition of curvature constraints. As a result, they advise agains the global imposition of such conditions. In their survey of flexible functional forms, Barnett and Serletis (2008) remark that imposing curvature can render some functional forms less flexible.

polynomials of order four and the parameterizations used to approximate them are processes of order two. Consequently, these second-order processes are not well-suited to approximate technologies whose price output set frontiers have pronounced curvature.[16]

Second, an increase in curvature of true frontiers often results in a decrease in the percentage of estimates which possess the `wrong' curvature as reported in Table 2. For the quadratic revenue function this result holds in both types of true models, whereas for the translog price output distance function it holds in true translog models only. For example, when K=100 and $g = (1,3)$ the proportion of the quadratic function-based estimates possessing wrong curvature falls from 52.3% to 16.4% and then to just 4% in Models P1B, P2B, and P3B, respectively. For the translog function-based estimates in samples of 100, the fraction of estimates with wrong curvature drops from about 25% (Model L1, column 8) to 12% (Model L3, column 10) in the true translog models, but stays roughly unchanged and extremely high in the case of polynomial technologies. This fall in the fraction of biased estimates when adding curvature is not unexpected, since adding curvature to the true frontier is likely to aid the estimation algorithm in obtaining a properly shaped estimate.

The plots in Figures 3 and 4 provide visual evidence of the variation in curvature of frontier estimates for samples of 100 observations. They also illustrate how the quadratic estimates of the true frontier perform compared to their translog counterparts. These representative empirical analogues were recovered using the parameter estimates of the quadratic revenue function and the translog distance function. Note that in the case of polynomial

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[16] This result is consistent with that of Guilkey et al. (1983), who found that when compared to the generalized Leontieff and the generalized Cobb-Douglas, the translog function "… provides a dependable approximation to reality provided that reality is not too complex."

technologies, depicted in Figure 3, both parameterizations usually yield incorrectly shaped estimates in models with little curvature, such as Models P1A and P1B. Adding more curvature seems to change the shape of the quadratic, but not the translog, estimates. This pattern appears in the case of the true translog technologies as well, whose corresponding plots are in Figure 4, with both the quadratic and translog estimates becoming concave in true models with a significant amount of curvature. In other words, while the shape of quadratic frontier estimates improves whenever more curvature is added to either type of true model, the translog parameterization seems to be doing a better job only when this true technology is translog, suggesting that the quadratic revenue function may be more flexible than the translog price output distance function.

Third, regardless of the sample size or the directional vector, quadratic functions perform better when approximating type-A models rather than their corresponding type-B counterparts, see Table 1, columns 2-7. Recall that the difference between these two subclasses is in the average size of price ratios, assumed to be closer to one in type-B models than in type-A models (low p_1, high p_2). In contrast, translog parameterizations appear to favor the type-B subclass of true polynomial models. This difference in the approximation quality appears to increase as more curvature is added to the true frontier. These results seem to suggest that the price output distance function dominates the directional revenue function when approximating polynomial models characterized by similar prices, especially when the corresponding true frontier is expected to be relatively flat. The directional revenue function performs better when the prices are relatively different, as well as when the true frontier has pronounced curvature.

Fourth, improvements in the quality of approximation resulting from an increase in sample size are more common among quadratic than among translog estimates for true polynomial models. For true translog technologies, both quadratic and translog estimates improve with sample size. Columns 2 through 7 of Table 1 indicate that in true polynomial models, the translog function's performance can deteriorate with an increase in the number of observations, and a similar drop in the quality of approximation often takes place in quadratic specifications that assume $g = (3,1)$. This contrasts with the remaining two directions, where sample size related improvements become more pronounced as the directional vector is rotated northward from $g = (1,1)$ to $g = (1,3)$, suggesting that the approximation quality of the quadratic function may be susceptible to the choice of the underlying directional vector.

The plots in Figure 5, which correspond to Model P3B and *K*=100, illustrate how the quadratic frontier's fit gradually deteriorates as the mapping vector is rotated from a mostly northern direction of expansion, such as $g = (1,3)$, toward a mostly eastern direction, like $g = (3,1)$. Indeed, the frontier estimate corresponding to $g = (3,1)$ appears to have the wrong curvature, whereas its counterpart in direction $g = (1,3)$ produces a reasonably close fit.

Recall that in order to introduce the price-space counterpart of 'technical inefficiency' and the disturbance term to our true models we chose to subtract the error components ν and ε directly from the second price. In the context of the quadratic revenue function this is equivalent to assuming a purely northern orthogonal directional vector, such as $g = (0,1)$. Consequently, we should expect the approximation quality to worsen as our assumptions regarding this vector progressively deviate from this orthogonality assumption, which underlies the data generating process. Unfortunately, this result is of relatively little practical value, since researchers have almost no a priori knowledge of the 'true' directional vector. Therefore, those who estimate directional revenue functions should consider assessing the robustness of their results to alternative direction vectors.

Last but not least, it appears that the quadratic parameterizations are overall better than translog in approximating both types of true technologies, polynomial and translog alike. Columns 2, 4, and 6 of Table 1 show that the quadratic function is unambiguously better than translog in the case of all type-A (low p_1, high p_2) polynomial technologies. This manifests itself in the form of lower mean benchmark values, as well as a smaller percentage of convex frontier estimates (Table 2). The fact the translog parameterization yields convex rather than concave frontier estimates in all but one type-A models over 99% of the time regardless of sample size is particularly disappointing. Perhaps this functional form may not be as flexible in certain situations as has been assumed in the literature. We note that this result contrasts with the results of previous simulation studies of flexible functional forms—neither Guilkey et al. (1983) nor Gagné and Ouellette (1998) managed to find a function with approximation properties superior to those of the translog.

On the other hand, the translog function's inherent tendency to produce convex boundaries can be very useful when the technology must be modeled using the cost function or the input distance function [Shephard (1970)], both of which are defined with respect to the input sets $L(y) = \{x : x \quad can \quad produce \quad y\}$. Relying on the translog to parameterize these functions can be recommended not only due to their homogeneity, but also because the frontier of input sets is assumed to be convex.[17]

Columns 3, 5, and 7 of Table 1, which contain the results corresponding to type-B models, imply that the quadratic function should be given preference only in large samples and

[17] The investigation of the relative quality of approximation in models that correspond to convex frontiers is left as a topic for future research. Recall that our analysis involves concave boundaries only.

for our case when the directional vector is predominantly northern. Although the quadratic function's dominance is not universal in this class of true models, we find that there exists at least one directional vector, such as $g = (1, 3)$ in the case of Models P2B and P3B, whose corresponding quadratic specification will outperform the translog. Nevertheless, the translog approximation outperforms other quadratic parameterizations for type-B technologies, especially those that assume mostly eastern directional vectors.

The rate at which the quadratic estimates first catch up with and then outperform their translog counterparts in type-B models accelerates with more curvature in the true model. For example, when $K = 500$ and $g = (1, 3)$ the difference between the mean benchmarks, which is consistently in favor of the quadratic function, grows from 0.108 in Model P1B $(0.505 - 0.397)$ to 0.299 and then 0.509 in Models P2B and P3B, respectively. At the same time, the fraction of frontier estimates that possess wrong curvature falls from 48.6 to zero among quadratic parameterizations but increases from an already high 90.5 to 99.6 percent in the case of the translog function (columns 3 and 7 of Table 2). To reiterate, the quadratic function's dominance is not universal and gradually disappears in type-B models with each eastward rotation of the directional vector—a result that is particularly noticeable in small samples.

Finally, the relative quality of approximation of true translog technologies is similar to that of type-B polynomial models. Mean benchmark discrepancies in the last three columns of Table 1 show that translog parameterizations dominate quadratic specifications in cases when the latter are based on mostly eastern directional vectors. As in the case of type-B models, the quadratic revenue function's approximation properties gradually improve as this vector is rotated northward, especially in large samples. Indeed, when $K = 500$ and $g = (1, 3)$ the quadratic function outperforms the translog in both Models L2 and L3, as well as produces fewer curvature violations than does the translog function.

To summarize, although the quadratic function does not appear to be a clear favorite, it usually beats the translog function in large samples and when the true frontier has pronounced curvature. Therefore, we recommend the use of both functional forms in preliminary stages of empirical studies. Since the quadratic frontier estimates seem to be sensitive to the choice of the directional vector, we recommend robustness checks with respect to the choice of direction vector.

6. Conclusions

Our goal is specification of revenue functions in their dual price space. We consider two distance functions: a Shephard output distance function and a directional output distance function, both defined in price space. Functional equation methods and properties of the distance functions provide some guidance on the choice of functional form for these: homogeneity in the Shephard case yields translog, translation in the directional distance function case yields a quadratic functional form. We employ Monte Carlo methods to assess the relative performance of these two functional forms.

Our simulation results regarding the revenue function representations in price space with Shephard and directional output distance functions are mixed. While the quadratic directional distance function generally outperforms the translog Shephard distance function in true models that have a polynomial structure, the opposite is usually true when the true technology is translog.

The quality of quadratic approximations appears to be sensitive to the choice of the directional vector. Some quadratic specifications outperform the translog regardless of the type of the true technology, especially in large samples. We have also encountered cases when the translog parameterization performed reasonably well, especially in small samples, although its ability to approximate frontiers with a relatively large amount of curvature was questionable.

In terms of our effort here with respect to the revenue function, we have provided evidence that translog specifications can sometimes yield imprecise estimates of technology, despite the fact that they satisfy the homogeneity property. This problem is particularly serious when the true technology is characterized by relatively unbalanced prices. Fortunately, in cases when translog parameterizations are inadequate, the quadratic directional output distance function in price space can be relied upon to provide an alternative way to identify the revenue function.

7. Appendix

Below we provide a brief sketch of the proof of the lemma. It is similar to Luenberger's (1995, p. 100) proof of the relation between the utility function and the benefit function.

Recall that the revenue function is convex in prices and thus continuous on the interior of $\Re_+^M$ (Shephard, 1970, p. 230). It is also non-decreasing in prices.

Following Luenberger (1995, p. 100), given that

i) $R(x, p+\alpha g) > R(x, p), \alpha > 0$ (a condition on R),

If $R(x, p) = R$ then $\Delta(x, p, R; g) = 0$.

Conversely, if $p \in$ Interior of $\Re_+^M$ and $\Delta(x, p, R; g) = 0$, then $R(x, p) = R$.

Details:

Assume that $R(x, p) = R$. Then $\Delta(x, p, R; g) \geq 0$. Since i) holds, $R(x, p+\alpha g) > R(x, p), \alpha > 0$. Thus, $\Delta(x, p, R; g) = 0$.

Conversely, let $p \in$ Interior of $\Re_+^M$. Then $\Delta(x, p, R; g) = 0$ implies that $R(x, p) \leq R$ and $R(x, p+\alpha g) > R(x, p)$. By continuity of R in p, $R(x, p) = R$.

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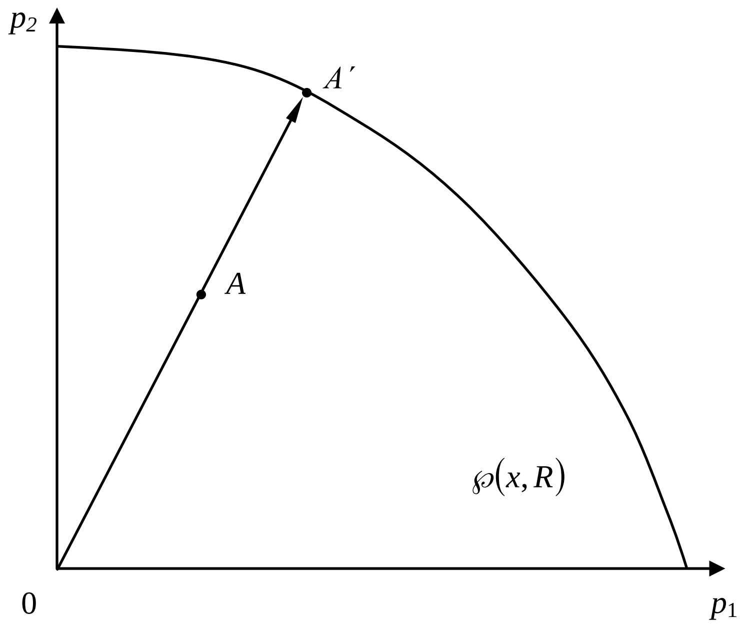
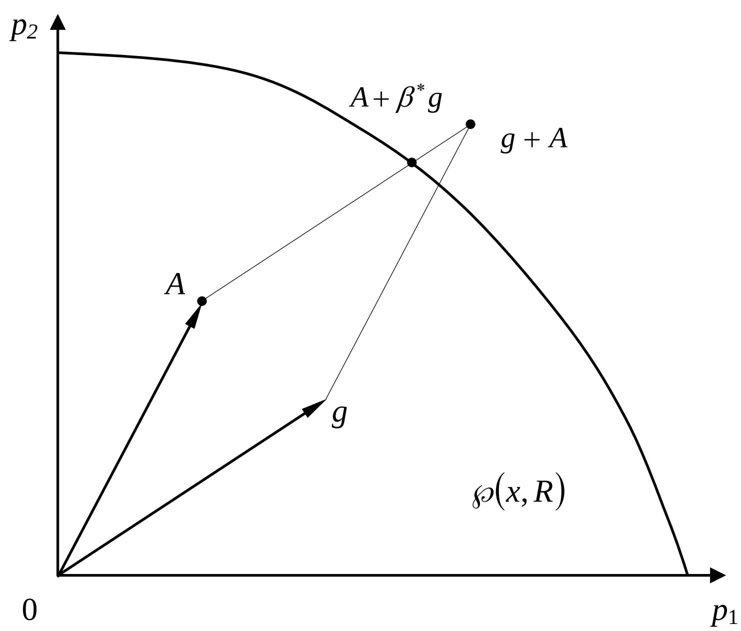
Figure 1

The Shephard Price Output Distance Function and the Directional Revenue Function

I. Shephard Price Output Distance Function $D(x, p, R)$

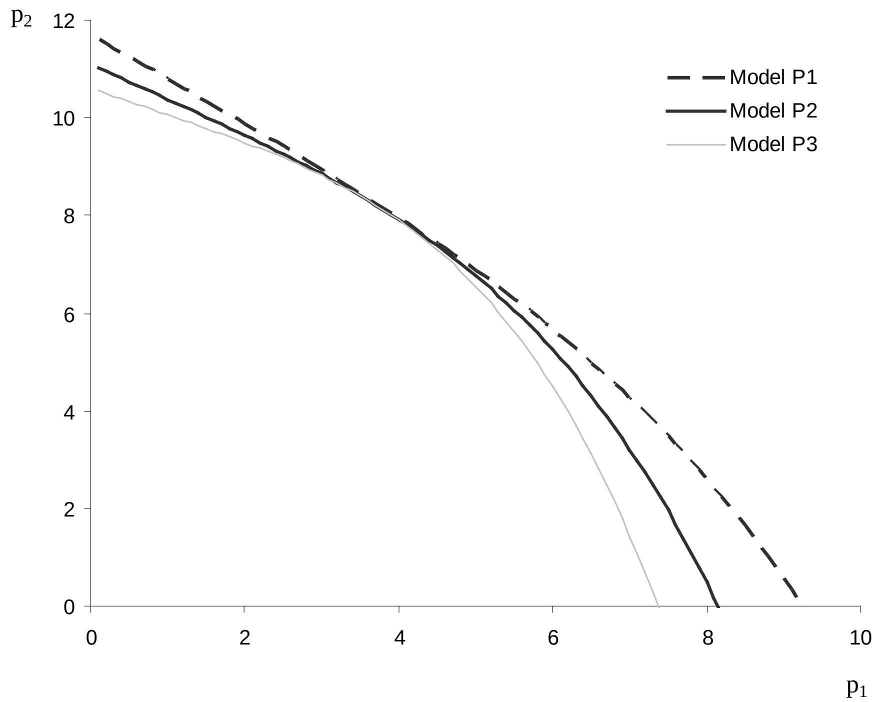


II. Directional Revenue Function $\Delta(x, p, R; g)$



<u>Figure 2</u>
True Frontiers of the Price Space Output Set

I. Polynomial Technologies



II. Translog Technologies

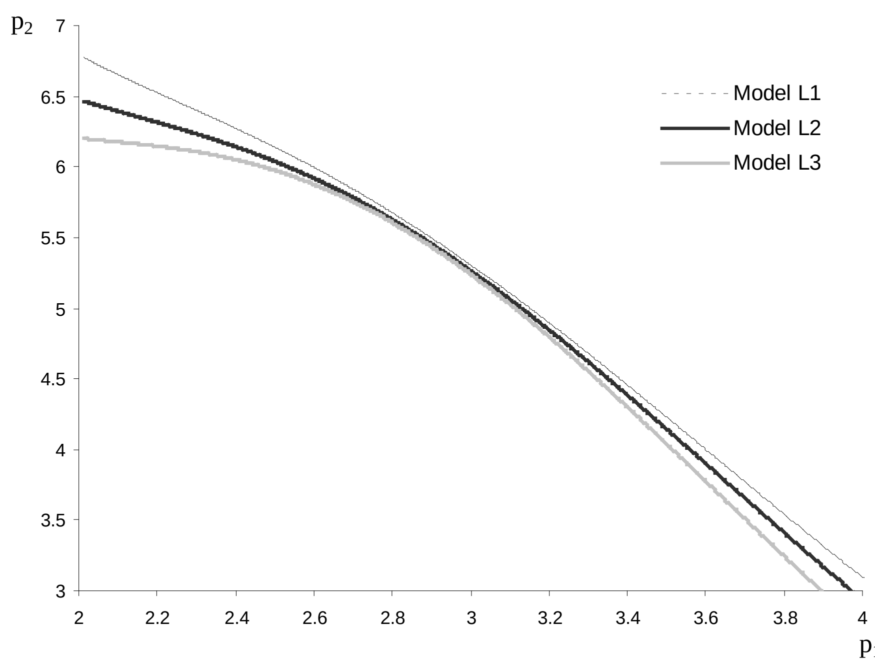


Table 1
Weighted Average Discrepancy between the True and Simulated Benchmark Values; Various Models

	Model P1A	Model P1B	Model P2A	Model P2B	Model P3A	Model P3B	Model L1	Model L2	Model L3
Quadratic Directional Revenue Function									
g = (3,1)									
K=50	0.678	0.783	0.805	1.024	0.983	1.283	1.615	1.981	2.718
K=100	0.680	0.797	0.805	1.007	1.012	1.229	1.452	1.868	2.627
K=500	0.703	0.777	0.864	0.980	1.057	1.174	1.296	1.711	2.501
g = (1,1)									
K=50	0.651	0.750	0.732	0.939	0.906	1.116	1.571	1.962	2.622
K=100	0.614	0.677	0.707	0.851	0.863	1.036	1.413	1.831	2.500
K=500	0.570	0.617	0.671	0.782	0.801	0.959	1.234	1.627	2.325
g = (1,3)									
K=50	0.647	0.679	0.682	0.749	0.747	0.849	1.464	1.742	2.390
K=100	0.465	0.500	0.508	0.571	0.556	0.706	1.263	1.564	2.181
K=500	0.356	0.397	0.401	0.503	0.461	0.625	1.086	1.382	2.010
Translog Price Output Distance Function									
K=50	0.958	0.577	1.282	0.714	1.524	0.973	1.189	1.520	2.219
K=100	0.804	0.463	1.111	0.659	1.424	1.007	1.060	1.429	2.157
K=500	0.797	0.505	1.168	0.802	1.572	1.134	0.986	1.423	2.179

Note: Recall that type-A technologies are associated with relatively low values of p_1 and relatively high values of p_2, whereas type-B technologies assume a relatively similar range for p_1 and p_2.

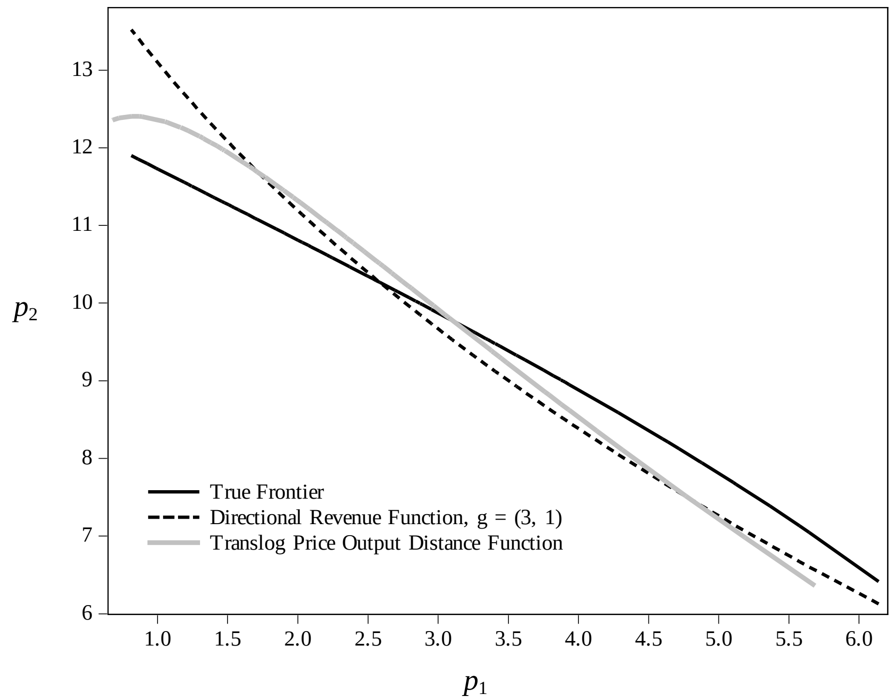
Table 2
Fraction of Frontier Estimates that Possess Wrong Curvature

	Model P1A	Model P1B	Model P2A	Model P2B	Model P3A	Model P3B	Model L1	Model L2	Model L3
Quadratic Directional Revenue Function									
g = (3,1)									
K=50	81.0	67.3	81.8	66.3	82.6	53.1	54.0	52.3	49.6
K=100	92.5	75.0	93.3	73.1	90.2	55.5	55.0	54.0	51.3
K=500	100.0	97.8	100.0	94.6	100.0	62.4	59.0	52.0	55.0
g = (1,1)									
K=50	83.4	63.6	78.6	48.5	72.6	34.3	47.8	42.4	40.7
K=100	94.4	71.4	88.4	54.4	80.2	24.3	48.0	43.2	36.1
K=500	100.0	93.2	100.0	47.8	95.8	2.8	44.2	28.4	19.6
g = (1,3)									
K=50	72.2	53.5	63.1	27.8	46.7	12.3	41.8	23.1	20.2
K=100	82.0	52.3	58.9	16.4	35.8	4.0	31.3	20.7	9.4
K=500	97.2	48.6	66.4	1.6	11.2	0.0	12.4	0.8	0.6
Translog Price Output Distance Function									
K=50	99.6	59.8	99.7	49.3	98.8	55.4	32.4	24.0	18.3
K=100	100.0	62.4	100.0	54.3	100.0	72.0	25.1	15.4	12.0
K=500	100.0	90.5	100.0	97.2	100.0	99.6	11.2	13.2	12.4

Figure 3

True and Simulated Frontiers of the Price Output Set; Polynomial Technologies

Model P1A



Model P1B

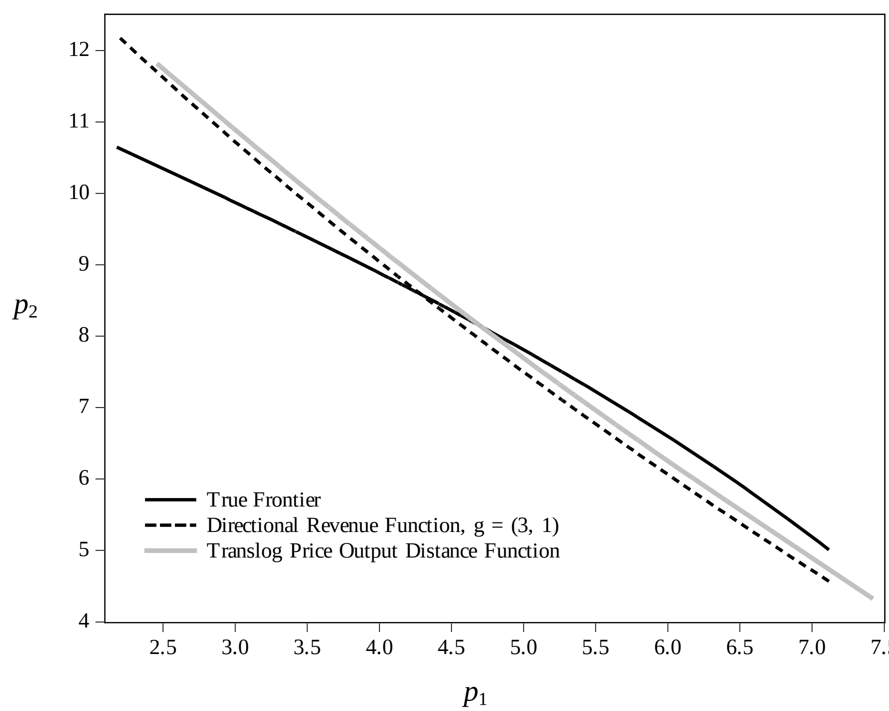
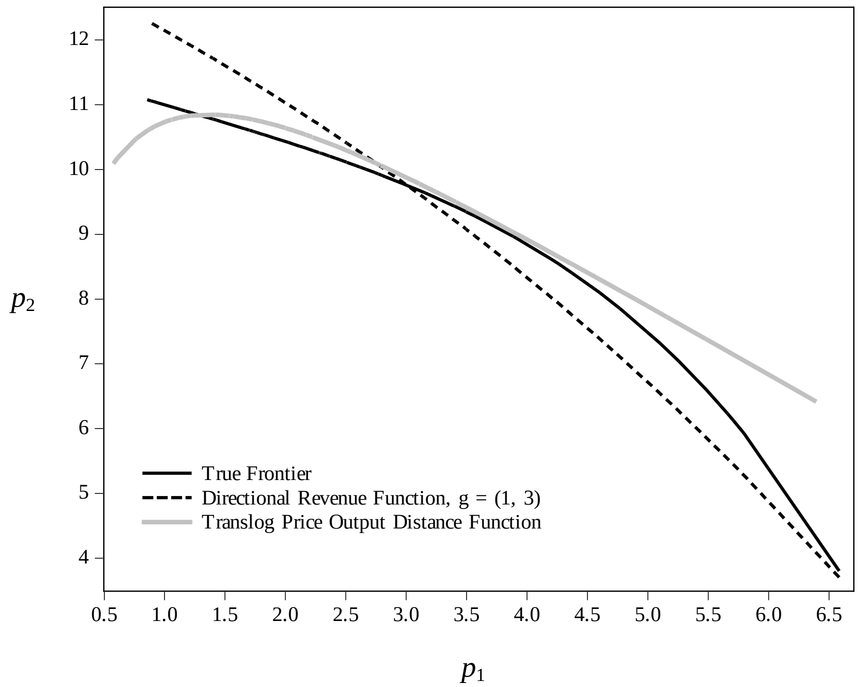


Figure 3 (continued)

Model P3A



Model P3B

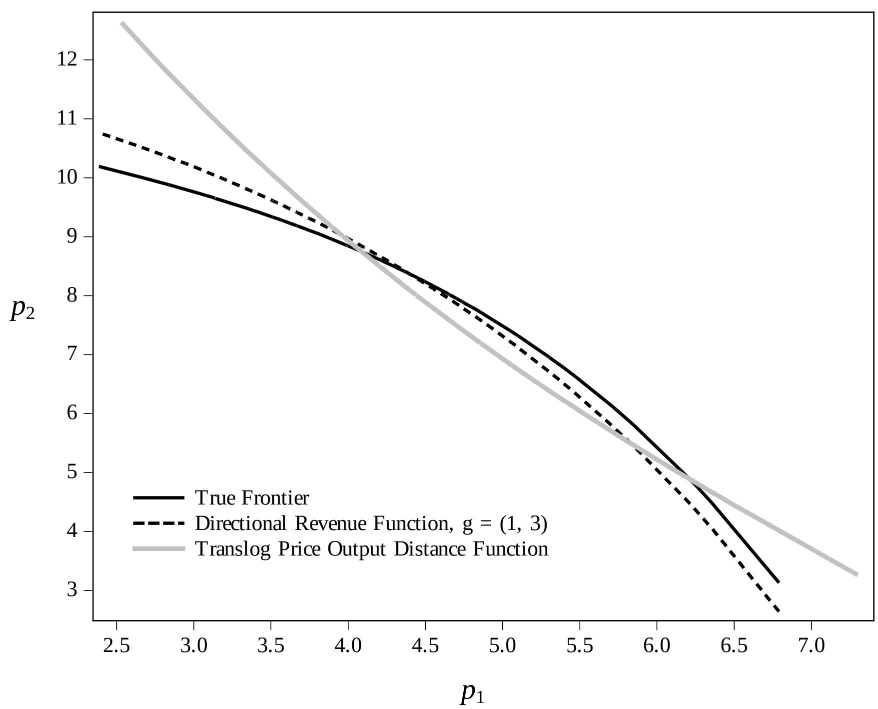
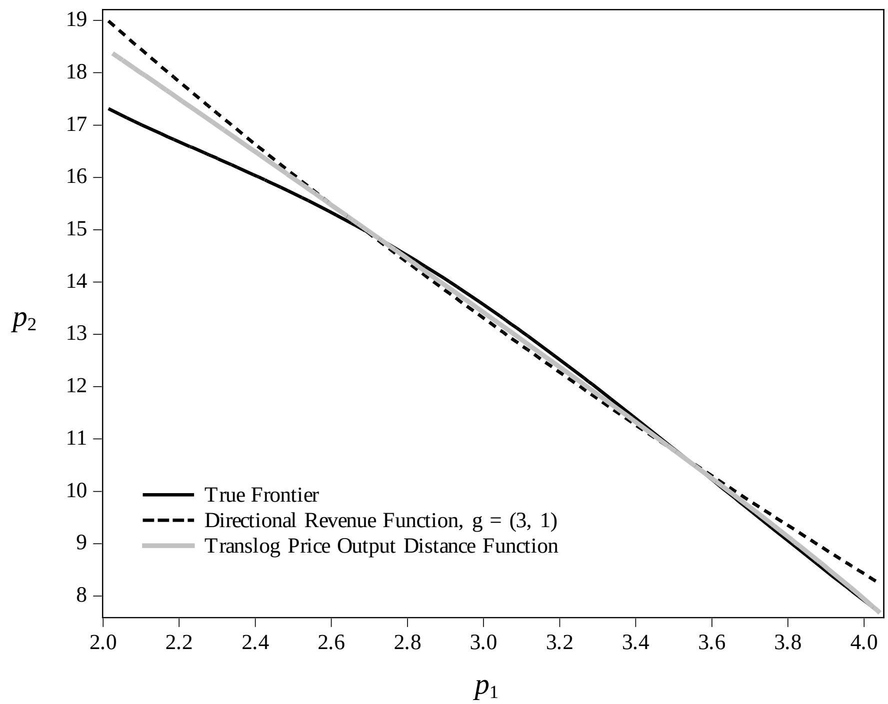


Figure 4
True and Simulated Frontiers of the Price Output Set; Translog Technologies

Model L1



Model L2

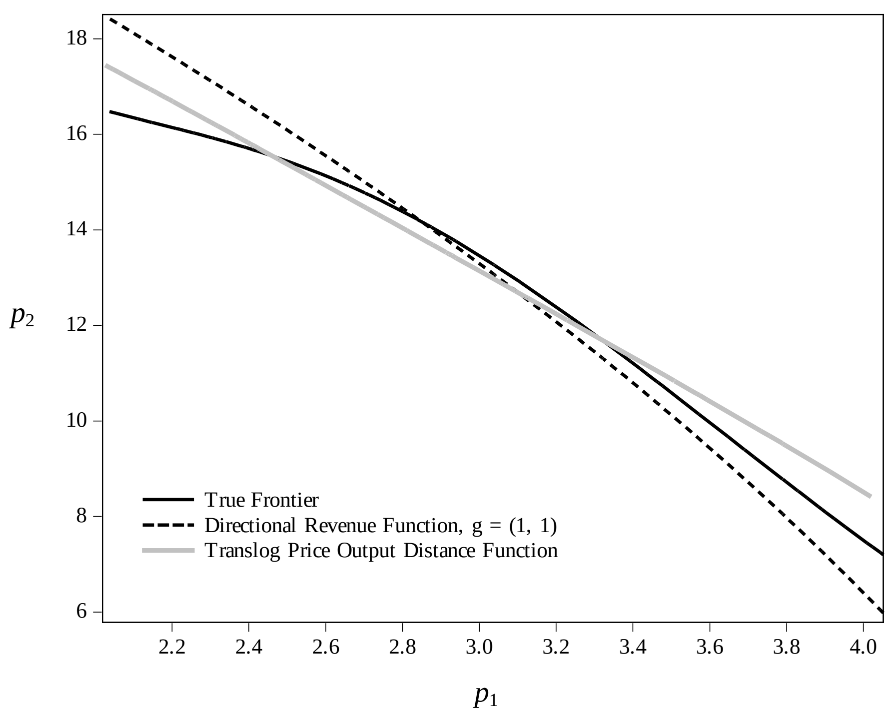
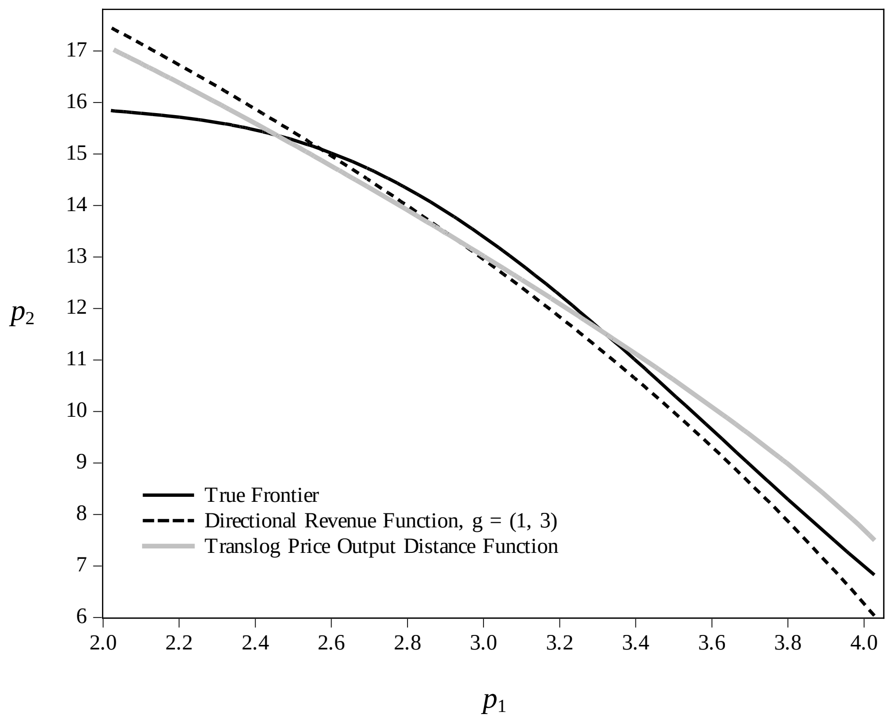


Figure 4 (continued)

Model L3



<u>Figure 5</u>
Model P3B Directional Revenue Function Estimates of the Price Output Set Frontier; Various Mapping Vectors