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Abstract approved ______________________________

William M. Stone

Methods for determining bounds for the probability of error in decoding messages sent over a continuous Gaussian channel have been investigated in papers by Claude E. Shannon and David Slepian of the Bell Telephone Laboratories. In these papers the problem is shown to reduce to finding the probability that a point on the surface of an n-dimensional spherical surface will be moved out of a polytope, apex at the origin and center line through the point. Using this model Shannon determined asymptotic bounds for $Q_n(\omega_1)$ and $\bar{Q}_n(\omega_1)$, the lower and upper bounds respectively for the decoding error probability, ω_1 the generating angle of the cone which replaces the polytope. Slepian determined an exact expression for $Q_n(\omega_1)$ in terms of a triple recursion formula. For the upper bound he obtained an integral of the function $Q_n(\omega) \sin^{n-2} \omega$ from 0 to ω_1. This integral had to be evaluated by a trapezoidal rule for 150 data points. In the present paper three alternative methods are given for determining the lower and upper bounds on the probability of error in decoding. The first two depend on the tabulations of Poisson's distribution and the Hh function. The third is an exact solution in terms of polynomials and exponentials in an integrand. Thus a significant simplification of the evaluation of the bounds is obtained.

On the Shannon-Slepian Estimates of Probability of Decoding Error

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Warren Gale Clendenin

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TABLE OF CONTENTS

ON THE SHANNON-SLEPIAN ESTIMATES OF PROBABILITY OF DECODING ERROR

INTRODUCTION

In 1959 Claude E. Shannon (12) of the Bell Telephone Laboratories published a paper describing a method for bounding the probability of decoding error for communication using an optimum code on a continuous Gaussian channel and determined asymptotic bounds for this error as a function of several system parameters. His results enable one to compare the merits of different codes or, given a communication system, to determine the coding of M words to minimize the probability of error in decoding. In 1963 David Slepian (13), also of the Bell Telephone Laboratory, extended the work of Shannon and obtained explicit expressions for the error bounds which he evaluated numerically. In this paper an alternate approach is used to obtain expressions for the error bounds which are at least different from those of Slepian and may well be somewhat easier to evaluate, particularly the upper bounds.

The "Shannon-Slepian" (12, 13) method of determining error bounds applies to M-word block codes of length n. That is, codes which map the integers 1, 2, ..., M onto the code words $m_1, m_2, \ldots, m_M$, where each word m_i is a sequence of n real numbers $s_{i1}, s_{i2}, \ldots, s_{iM}$. The integers 1, 2, ..., M represent words in a code book, not to be confused with the code words m_i. The codes to be used are also restricted by a transmitter power limitation. In general, each code word is constrained to be transmitted with the same signal power. The method, however, may be modified to allow codes where the code words are constrained to have a maximum or average signal power limitation.

It is further assumed by Shannon (12) that the code words have equal probability of occurrence and are transmitted over a channel having Gaussian noise which affects each component s_{ij} of the word m_i independently of all other components s_{ik}, $k \neq j$, the variance being N for each component. In other words, if the component s_{ij} is transmitted some other real number $s_{ij} + x_j$ will be received, where each x_j is an independent Gaussian random variable with variance N.

Code words received at the receiver are decoded according to an optimal decoding system. That is, there exists a geometric criterion such that the probability of error in decoding is minimized. The representation of the code words as a sequence of n real numbers suggested to Shannon (12) that he interpret each code word as a message point, or a message vector, in n-dimensional space, where the s_{ij} represent the coordinates of the i-th message point. Adoption of Shannon's (12) geometrical viewpoint allows us to visualize coding and decoding systems and aids us in analyzing these systems. A decoding system is a partitioning of n-dimensional space into M subsets, each corresponding to one of the M integers. Decoding is accomplished by assigning the value of i corresponding to the subset of n-dimensional space into which a received message point falls to the message. An error in decoding occurs when the value of the index assigned differs from that of the word actually transmitted.

The geometrical interpretation of the channel noise is that of a noise vector centered at the transmitted message point in n-dimensional space. As remarked above, each of the n components has an independent Gaussian distribution with variance N. The effect of this noise vector is that of an additive vector which moves the endpoint of the message vector to a new point in space. It is this new point that is received and decoded. Therefore, an error in decoding occurs when the noise vector takes a message point outside its assigned volume of n-dimensional space.

If distance in n-dimensional space corresponds to signal amplitude we can also express the limitation on the transmitter power in geometrical terms. Signal power is proportional to the square of the signal amplitude; therefore, requiring each code word to be transmitted with the same power is equivalent to requiring that each code word have the same absolute magnitude, or, geometrically, that each message point lies at the same distance from the origin. If we let nP be the transmitter power we then have the requirement that all message points lie on the surface of an n-dimensional ball of radius $(nP)^{\frac{1}{2}}$.

Let us now consider the distribution function of the noise in this geometrical coding scheme. Let x_j be the noise component along the j-th coordinate axis, $j = 1, 2, \ldots, n$. Each probability density has the well-known Gaussian form,

$$p_{x_j}(a_j) = \frac{1}{(2\pi N)^{\frac{1}{2}}} e^{-\alpha_j^2/2N}, \quad -\infty < \alpha_j < \infty, \tag{1.1}$$

and, because of statistical independence, the vector valued random variable has the spherical form of the density,

$$p_{\bar{x}}(\bar{\alpha}) = \frac{1}{(2\pi N)^{n/2}} e^{-|\bar{\alpha}|^2/2N} , \tag{1.2}$$

where $|\bar{\alpha}|^2 = \alpha_1^2 + \alpha_2^2 + \ldots + \alpha_n^2$. For the random variable taken to be r, the Euclidean distance measured from the message point, the density function of interest in later developments has the normalized form,

$$p_r(\alpha) = \frac{1}{2^{(n-3)/2}\Gamma(\frac{n-1}{2})} \alpha^{n-2} e^{-\alpha^2/2}, \ \alpha > 0, \ n \geq 2. \tag{1.3}$$

Noting that the spherical Gaussian distribution function is monotone decreasing with distance from the origin, and that the origin of the noise vector is a message point, we see that the probability of a message point being moved a distance r from the origin decreases with increasing r. Therefore, we can minimize the probability of error in decoding, or devise an optimal decoding scheme, by decoding a received message point as the integer corresponding to the message point nearest it in n-dimensional space. Such a system is known as minimum distance decoding or maximum likelihood decoding. Geometrically it requires the partitioning of n-dimensional space into M n-dimensional polytopes whose sides are the $n-1$ dimensional hyperplanes which are the perpendicular bisectors of the set of $\binom{n}{2}$ chords required to connect each message point with all the other message points. As each message point is at the same distance from the origin of the coordinate system each polytope will be a symmetric n-dimensional pyramid, apex at the origin and axis of symmetry passing through the origin and the message point.

This geometric approach to the coding and decoding problem reduces the problem of determining the probability of decoding error to determining the probability that a message point will be moved outside its assigned pyramid by the noise vector.

Let us now consider the problem of placing a lower bound on the probability of decoding error, as presented by Shannon (12). Let P_e be the probability of decoding error for the code and let P_{ei} be the probability of decoding error when the i-th word is transmitted. Remembering that each word is equally likely to be selected for transmission we then have

$$P_e = \frac{1}{M} \sum_{i=1}^{M} P_{ei} \; . \tag{1.4}$$

Using our geometrical approach P_{ei} is the probability that the i-th message point will be moved by noise outside its pyramid. Let the i-th pyramid have the solid angle W_i, where the solid angle in n-dimensional space is equivalent to the area cut out of an origin-centered spherical surface of unit radius by the pyramid. We can now replace the pyramid by a cone of spherical cross-section of degree $n-1$, same apex and line of symmetry as the pyramid. For $n = 3$ it would be the familiar cone of circular cross section. The probability that the message point will be removed from the cone by the effect of the noise can be shown to be less than the probability that it will be so removed from the pyramid, so we can bound the probability of decoding error from below by studying the easier problem posed by the cone. We can establish this rather simply in the three-dimensional case depicted in Figure 1;

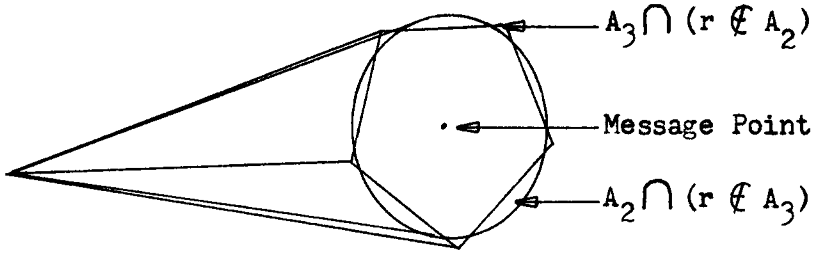

Fig. 1.

A simplified discussion is available in Reza (10, p. 325-327); let $p_r(\alpha)$ represent a two-dimensional, monotonically decreasing probability density, and compare the probability of the randomly placed point, distant r units from the center of a circle, falling within that circle with the probability of falling within a polygon curve which encloses the same area as the circle. Let A_1 be the common area enclosed, A_2 the area enclosed by the circle, A_3 the area enclosed by the polygon curve. Then

$$P[r \in A_2] = P[r \in A_1] + P[r \in A_2 \cap (r \notin A_3)] \ . \tag{1.5}$$

Consider now elements of equal area, one within the polygon curve but not within the circle, the other element within the circle but not within the polygon curve. Because of the monotone nature of $p_r(\alpha)$

we conclude that there is a smaller probability associated with the event that the point falls into the former element. Thus

$$P[r \in A_3 \cap (r \notin A_2)] \leq P[r \in A_2 \cap (r \notin A_3)] . \tag{1.6}$$

The same will hold true for a cross-section taken perpendicular to the axis of symmetry at any distance from the origin, so the probability that a message point moved at random will remain within the cone so constructed is greater than the probability that it will remain within the pyramid. Hence, the probability of error in decoding, P_{ei}, is greater than $Q^*(W_i)$, which is the probability that the i-th message point be removed from its assigned conical region in 3-space. The results hold true for the n-dimensional case, noting that the cross-section would be an n-1 dimensional figure. We now have

$$P_e \geq \frac{1}{M} \sum_{i=1}^{M} Q^*(W_i) . \tag{1.7}$$

As the pyramids cover all of the n-dimensional space we also have

$$\sum_{i=1}^{M} W_i = W_o = W(\pi) , \tag{1.8}$$

where W_o is the solid angle associated with the n-dimensional spherical surface.

We can simplify this bound further by observing that, as the density function decreases with distance, $Q(W)$ is a concave function of W. Referring to Figure 2, which indicates the behavior of $Q^*(W)$, and noting that there are M values of W_i which sum to W_o, we see that for each value of W_i less than W_o/M, $Q(W_i)$ is greater than $Q(W_o/M)$. For each value W_j of W greater than W_o/M there exists at least one value W_k less than W_o/M. If we replace both W_j and W_k by

$$W_j' = W_k' = \frac{W_j + W_k}{2} \tag{1.9}$$

and replace both $Q^*(W_j)$ and $Q^*(W_k)$ by

$$Q^{*\prime}(W_j) = Q^{*\prime}(W_k) = \frac{Q^*(W_j) + Q^*(W_k)}{2} \tag{1.10}$$

we do not change the values of the sums of (1.7) and (1.8). From Figure 2, however,

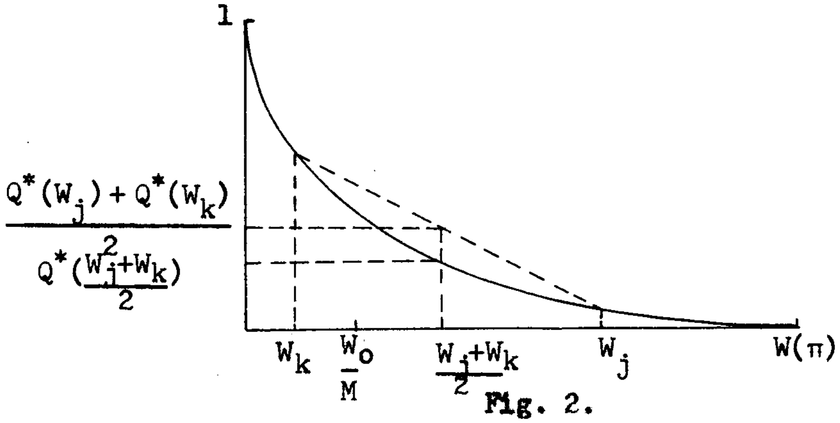

Fig. 2.

we see that

$$\frac{Q^*(W_j) + Q^*(W_k)}{2} \geq Q^*(\frac{W_j + W_k}{2}) \, , \tag{1.11}$$

so

$$P_e \geq \frac{1}{M} \sum_{i=1}^{M}{}' \; Q^*(W_i) + \frac{1}{M} Q^*\left(\frac{W_j + W_k}{2}\right) , \tag{1.12}$$

where the prime on the summation symbol means that the index i is not assigned the values j and k. Repeating this process for each value of $W_i > W_o/M$, including values of W_j and W_k, we will obtain finally the sums

$$\sum_{i=1}^{M} W_i^* = W_o , \tag{1.13}$$

where $W_i^* = W_o/M$ for all i, and

$$P_e \geq \frac{1}{M} \sum_{i=1}^{M} Q^*(W_o/M) = Q^*(W_o/M) . \tag{1.14}$$

If we now define ω_1 to be the half angle of the cone of solid angle W_o/M we can define $Q^*(W) = Q(\omega)$ and

$$P_e \geq Q(\omega_1) . \tag{1.15}$$

This will be our fundamental lower bound for the decoding error.

To place an upper bound on the probability of decoding error we will consider an ensemble of random codes, where each code in the ensemble is defined by placing M message points on the surface of an n-dimensional ball of radius $(nP)^{\frac{1}{2}}$. Each point in a code will be placed independently of all others and with probability measure proportional to the solid angle W_i, defined as above. We now observe that there must exist at least one code in the ensemble whose probability of decoding error is less than the average probability of decoding error taken over the ensemble, and, therefore, the optimum decoding system must yield an error at least as small. Due to the statistical independence of each message point in a code we find that the ensemble average probability of error is just M times the average probability for any one message point. Let us, therefore, consider the probability that the i-th message will be incorrectly decoded. The probability that this message will be selected and transmitted is $1/M$ and the probability that the message point will be moved into the region between cones of half angle ω and $\omega + d\omega$ is $-dQ(\omega)$, where the differential is itself negative and ω and $Q(\omega)$ are defined as above. An error in decoding will occur when a cone of half angle ω about this <u>received</u> point contains one or more message points. Remembering that each message point is placed with probability measure proportional to the solid angle, we see that the probability that any message point other than the i-th will be found in this received point cone is

$$P_{ej} = W(\omega)/W(\pi) \ , \ j \neq i \ , \tag{1.16}$$

and the probability that no such point will be found in the received point cone is $1 - [1 - \frac{W(\omega)}{W(\pi)}]^{M-1}$. Averaging over all possible noise displacements we obtain for the average probability of decoding error for the i-th message point

$$P_{ei} = -\frac{1}{M}\int_0^{\pi}\left(1 - \left[1 - \frac{W(\omega)}{W(\pi)}\right]^{M-1}\right) dQ(\omega) \ . \tag{1.17}$$

and for the ensemble of codes,

$$P_{er} = -\int_0^{\pi}\left(1 - \left[1 - \frac{W(\omega)}{W(\pi)}\right]^{M-1}\right) dQ(\omega) \ . \tag{1.18}$$

Note that $\left[1 - \frac{W(\omega)}{W(\pi)}\right]^{M-1} \leq 1$; the well-known inequality $(1-x)^n \geq 1 - nx$ may be written in the form $1 - (1-x)^n \leq nx$, so

$$1 - \left[1 - \frac{W(\omega)}{W(\pi)}\right]^{M-1} \leq M\frac{W(\omega)}{W(\pi)} \ . \tag{1.19}$$

and we obtain (recall that dQ is negative)

$$P_{er} \leq -\int_0^{\omega_1} M\frac{W(\omega)}{W(\pi)}\, dQ(\omega) - \int_{\omega_1}^{\pi_1} dQ(\omega) \ . \tag{1.20}$$

In the second interval of integration the original integrand was simply bounded by unity. This gives us

$$P_e \leq P_{er} \leq Q(\omega_1) - \frac{M}{W(\pi)}\int_0^{\omega_1} W(\omega)\, dQ(\omega) \tag{1.21}$$

for our fundamental upper bound. Complete inequality stands as

$$Q(\omega_1) \leq P_e \leq Q(\omega_1) - \frac{M}{W(\pi)} \int_0^{\omega_1} W(\omega)\, dQ(\omega) \ . \tag{1.22}$$

We now need to express these bounds in terms of the system parameters, n the dimensionality, M the number of signals transmitted, N the noise power, P the signal power. As a first step let us determine an expression for ω, the half angle of the cone which cuts out the "area" W on the surface of the n-dimensional unit ball. The surface of a ball of radius R in n-dimensional space is given as $\frac{n\, \pi^{n/2} R^{n-1}}{\Gamma(\frac{n}{2}+1)}$, a formula which checks out easily for n taken to be 2 and 3. The cross-section of a cone in n-dimensional space is an (n-1)-dimensional surface. To calculate the solid angle of the cone of half angle ω we can sum by integration the contributions to the area cut out on the unit n-dimensional spherical surface by the cone of (n-1)-dimensional rings of width $d\omega$. The area to be found is of $n-1$ dimensions so that a ring in $n-1$ dimensions is defined by an (n-2)-dimensional figure which is the surface of an (n-1)-dimensional ball. The differential area of the surface of the unit n-dimensional ball is, therefore, given (see Fig. 3) by a ring of radius $\sin \omega$ and width $d\omega$,

$$dW = \frac{(n-1)\pi^{\frac{n-1}{2}}}{\Gamma\frac{(n+1)}{2}} \sin^{n-2} t \ dt \ . \tag{1.23}$$

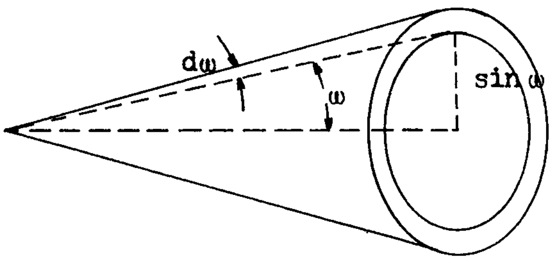

Fig. 3.

Integration yields

$$W(\omega) = \frac{(n-1)\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n+1}{2})} \int_0^{\omega} \sin^{n-2} t \, dt \; ; \tag{1.24}$$

substituting $\sin t = u$, $dt = \dfrac{du}{2[u(1-u)]^{\frac{1}{2}}}$, we have

$$W(\omega) = \frac{(n-1)\pi^{\frac{n-1}{2}}}{2\Gamma(\frac{n+1}{2})} \int_0^{\sin^2\omega} u^{\frac{n-3}{2}} (1-u)^{-\frac{1}{2}} \, du \; . \tag{1.25}$$

It is at once clear that

$$\frac{W(\omega)}{W(\pi)} = \frac{1}{2B(\frac{n-1}{2}, \frac{1}{2})} \int_0^{\sin^2\omega} u^{\frac{n-3}{2}} (1-u)^{-\frac{1}{2}} \, du = \frac{1}{2} I_{\sin^2\omega}(\frac{n-1}{2}, \frac{1}{2}) \; , \tag{1.26}$$

the well-tabulated Incomplete Beta Function of Pearson (1,9). The angle ω_1 is then given by the relationship

$$M = \frac{2}{I_{\sin^2\omega_1}(\frac{n-1}{2}, \frac{1}{2})} . \tag{1.28}$$

To determine the lower bound $Q(\omega_1)$ recall that $Q(\omega)$ is the probability that a noise vector with distribution function given by (1.2) and centered at a message point will move the message point out of a cone of half angle ω. Let us define a coordinate system in n-dimensional space with origin at a message point. Define the axis of symmetry of the cone as the y-axis and a radius vector perpendicular to the y-axis as the x-axis. Normalize these variates so the radius of the spherical surface on which the M message points are uniformly scattered is $(n)^{\frac{1}{2}}A$, where $A = (P/N)^{\frac{1}{2}}$. The pertinent noise density function now has the form

$$p_{x,y}(\alpha,\beta) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{x^2+y^2}{2}} . \tag{1.29}$$

As x is rotated about the y-axis it sweeps out an (n-1)-dimensional ball and the end of the x vector travels over an (n-1)-dimensional surface, so that an element of area $dxdy$ at a distance x from the y-axis will sweep out an n-dimensional volume

$$dv = \frac{(n-1)\pi^{\frac{n-1}{2}} x^{n-2}}{\Gamma(\frac{n+1}{2})} \, dxdy . \tag{1.30}$$

The probability element that a message point will be found in this volume after being moved by noise is then just

$$dP = \frac{(n-1)\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n+1}{2})} x^{n-2} \frac{1}{(2\pi)^{n/2}} e^{-\frac{x^2+y^2}{2}} dxdy \; ; \qquad (1.31)$$

integration over the volume of n-space outside the cone yields

$$Q(\omega) = \frac{1}{(2\pi)^{\frac{1}{2}} 2^{\frac{n-3}{2}} \Gamma(\frac{n-1}{2})} \int_0^{\infty} dx \int_{-\infty}^{x\alpha-h} dy \; x^{n-2} e^{-\frac{x^2+y^2}{2}} , \qquad (1.32)$$

where $\alpha = \cot \omega$ and $h = (n)^{\frac{1}{2}} A$.

For the upper bound of the probability of error in decoding we have

$$P_e \leq \overline{Q}(\omega_1) = Q(\omega_1) - \frac{M}{W(\pi)} \int_0^{\omega_1} W(\omega) \; dQ(\omega)$$

$$= \frac{M}{W(\pi)} \int_0^{\omega_1} Q(\omega) \; dW(\omega) \; . \qquad (1.33)$$

From (1.23) and the formula for the surface of a unit ball in n-space (i.e., $W(\pi)$) the upper bound relationship takes the form

$$P_e \leq \frac{M}{(\pi)^{\frac{1}{2}}} \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})} \int_0^{\omega_1} Q(\omega) \sin^{n-2} \omega \; d\omega \; . \qquad (1.34)$$

Equations (1.32) and (1.34) allow us to express our upper and lower bounds in terms of the system parameters n, M, N and P. Equation (1.24) is the same as Shannon's (12) equation (21) and is obtained in the same way. Equations (1.28), (1.32) and (1.33) are the same as Slepian's (13) equations (7), (9) and (13), and are obtained in the same way.

For later comparison we will list and briefly describe the asymptotic bounds obtained by Shannon (12). The detailed analysis involved in obtaining these results is not presented. To quote Shannon, "It might be said that the algebra involved is in several places unusually tedious" (12, p. 615). Several expressions occur in these results which have not been previously defined (recall that the signal to noise power ratio is $A = (P/N)^{\frac{1}{2}}$):

$$\omega_0 = \cot^{-1} A \,,$$

$$G = G(\omega) = \frac{1}{2}\left(A\cos\omega + \sqrt{A^2\cos^2\omega + 4}\,\right) ,$$

ω_c = the solution of the equation $2\cos\omega - AG(\omega)\sin^2\omega = 0$,

$$E_L(\omega) = \frac{A^2}{2} - \frac{1}{2}AG\cos\omega - \log(G\sin\omega) \,.$$

Shannon (12) derives an equation comparable to (1.32) with integration over the variables ω and r , where $r = y/\cos\omega$. After much algebra he obtains

$$\frac{d}{d\omega}Q(\omega) = \frac{(n-1)\, e^{-\frac{h^2\sin^2\omega}{2}} \sin^{n-2}\omega}{2^{\frac{n}{2}}(\pi)^{\frac{1}{2}}\,\Gamma(\frac{n+1}{2})} \int_0^\infty r^{n-1} e^{-\frac{(r-h\cos\omega)^2}{2}}\, dr \,. \quad (1.35)$$

The integral in (1.35) is evaluated asymptotically, using a Lemma due to David and Kruskal (5), so

$$\frac{d}{d\omega}Q(\omega)\sim\frac{n-1}{(n\pi)^{\frac{1}{2}}}\;\frac{1}{(1+G^2)^{\frac{1}{2}}\sin^2\omega}\left[G\sin\omega\; e^{-\frac{A^2}{2}+\frac{AG}{2}\cos\omega}\right]^n, \qquad (1.36)$$

a result which requires the asymptotic expression for $\Gamma(\frac{n+1}{2})$. A simplified expression is

$$\frac{d}{d\omega}Q(\omega)\sim\alpha(\omega)\,e^{-nE_L(\omega)}. \qquad (1.37)$$

Shannon (12) then shows that for $\omega=\omega_o$ both $E_L(\omega)$ and $E_L'(\omega)=0$. Using this he shows that for $\omega_o\geq\omega\geq 0$ $dQ/d\omega$ is maximum at $\omega=\omega_1$. Using this and arguments involving the behavior of $\alpha(\omega+n^{-2/3})$ and $E_L(\omega+n^{-2/3})$, as well as a Taylor series expansion of $\exp(-nE_L(\omega))$, he shows that the sum of the integrations of $dQ/d\omega$ over the ranges ω_1 to $\omega_1+n^{-2/3}$, $\omega_1+n^{-2/3}$ to $\frac{\pi}{2}$, $\frac{\pi}{2}$ to π, which defines $Q(\omega_1)$, yields the asymptotic expression (replace ω_1 by ω)

$$Q(\omega)\sim\frac{\alpha(\omega)}{(n)^{\frac{1}{2}}}e^{-nE_L(\omega)}=\frac{e^{-nE_L(\omega)}}{(n\pi)^{\frac{1}{2}}(AG\sin^2\omega-\cos\omega)(1+G^2)^{\frac{1}{2}}\sin\omega}. \qquad (1.38)$$

Using the change of variable, $x=\sin\omega$, the mean value theorem and the asymptotic expression for $\Gamma(n)$, Shannon (12) obtains the asymptotic expression

$$\frac{W(\omega_1)}{W(\pi)} \sim \frac{\sin^n \omega_1}{(2n\pi)^{\frac{1}{2}} \sin \omega_1 \cos \omega_1} . \tag{1.39}$$

Substituting this expression and the expression for $\frac{d}{d\omega} Q(\omega)$ into (1.33) he obtains a sum of $Q(\omega)$ and a complicated integral in terms of n, P, N, G, ω for $\bar{Q}(\omega)$. Following the procedures used for evaluating $Q(\omega)$ he finds that the behavior of $\bar{Q}(\omega)$ depends on whether $\omega_1 \leq \omega_c$ or $\omega_1 \geq \omega_c$, the exponential term in the integral having a maximum at ω_c. For $\omega_1 \leq \omega_c$ the integral has a maximum at ω_1 and is evaluated in the same way as $Q(\omega)$, to give

$$\bar{Q}(\omega_1) \sim \frac{\alpha(\omega_1)}{(n)^{\frac{1}{2}}} \left(1 - \frac{\cos \omega_1 - AG \sin^2 \omega_1}{2 \cos \omega_1 - AG \sin^2 \omega_1}\right) e^{-nE_L(\omega_1)} . \tag{1.40}$$

For the case where $\omega_1 > \omega_c$ the integral is expressed as a sum of the integrals over the ranges 0 to $\omega_c - n^{-2/5}$, $\omega_c - n^{-2/5}$ to $\omega_c + n^{-2/5}$, and $\omega_c + n^{-2/5}$ to ω_1. Using essentially the same arguments as for the integral in $Q(\omega)$ Shannon (12) shows that this integral is asymptotic to $\dfrac{\exp(-nE_L(\omega_o) - nR)}{\cos \omega_c \sin^3 \omega_c \left[n\pi E_L(\omega_c)[1+G^2(\omega_c)]\right]^{1/2}}$, where

$$e^{nR} = M = \frac{W(\pi)}{W(\omega_1)} \tag{1.41}$$

and R is defined as the signal rate for the code and has units of decimal digits per dimension. Comparing the exponents in this expression and in the expression for $Q(\omega)$ Shannon (12) shows that

$$\bar{Q}(\omega_1) \sim \frac{M\, e^{-nE_L(\omega_c)}}{\cos\omega_c \sin^3\omega_c \left(n\pi E_L(\omega_c)[1+G^2(\omega_c)] \right)^{1/2}} \qquad (1.42)$$

In addition to asymptotic bounds Shannon (12) gives firm upper and lower bounds, obtained by consistently overbounding or underbounding the expression obtained in evaluating $\bar{Q}(\omega_1)$ and $Q(\omega_1)$. They are, for all values of n,

$$P_e \geq \frac{(n-1)^{\frac{1}{2}}\, e^{3/2}\, e^{-nE_L(\omega_1)}}{6n(A+1)^3\, e^{(A+1)^2/2}}\ , \qquad (1.43)$$

and if the maximum value of $G^n(\omega)\sin^{n-2}\omega\, e^{-\frac{n}{2}(A^2-AG\cos\omega)}$ is at $\omega = \omega_1$,

$$P_e \leq \omega_1 (2n)^{\frac{1}{2}}\, e^{3/2}\, G^n(\omega_1)\sin^{n-2}\omega_1\, e^{-\frac{n}{2}(A^2-AG\cos\omega_1)} \cdot \left[1 + \frac{1}{n\omega_1 \min[A, AG(\omega_1)\sin\omega_1 - \cot\omega_1]} \right] . \qquad (1.44)$$

For $\omega_1 > \omega_c$ Shannon (12) also obtains the bounds

$$P_e \leq \frac{1}{\lambda A (n\pi)^{\frac{1}{2}}}\, e^{n[R - \frac{\lambda^2 A^2}{4}]}\ . \qquad (1.45)$$

where

$$R = (1-\frac{1}{n})\log \frac{1}{\sin(2\sin^{-1}\frac{\lambda}{(2)^{\frac{1}{2}}})} , \tag{1.46}$$

and

$$P_e \geq \frac{1}{2}\ \varphi[-A(\frac{2M}{2M-1}\ \frac{n}{2})^{\frac{1}{2}}] , \tag{1.47}$$

where $\varphi(x)$ is the normal distribution function with unit variance.

For ω_1 near ω_o, $\omega_1 < \omega_c$, the asymptotic bounds are very close, either one giving a good approximation in the error of decoding. For ω_1 near ω_c, however, the bounds diverge. For $\omega_1 > \omega_c$ and $R \to 0$, $\lambda \to 1$, the firm upper and lower bounds are nearly equal for large n, giving a good approximation for the error of decoding.

These results may be expressed in terms of the signal rate R and the channel capacity C, where

$$C = \frac{1}{2}\log(A^2 + 1) . \tag{1.48}$$

The channel capacity is an upper bound for the signal rate. For rates less than C arbitrarily small probability of decoding error may be obtained for large enough n. For rates greater than C the probability of decoding error increases to unity with increasing n (11). For rates near the channel capacity such that $R < C$ both the upper and lower asymptotic bounds approach

$$P_e \sim \varphi\left[\left(n \frac{2P(P+N)}{N(P+2N)}\right)^{\frac{1}{2}}(R-C)\right] \tag{1.49}$$

for large values of n.

These results enable us to estimate the probability of decoding error when the code rate R is less than channel capacity C and when n is large. No estimate is given by Shannon (12) for how good the approximations are, particularly for small values of n. In order to determine the ranges of validity for the approximation and to give more accurate bounds David Slepian (13) obtained exact expressions for $Q(\omega_1)$ and $\bar{Q}(\omega_1)$. By integrating once by parts in (1.32) he arrived at the recursion

$$\frac{Q_n(\omega)}{c_n} = (n-3)\ \frac{Q_{n-2}(\omega)}{c_{n-2}} + \alpha J_{n-2}\ , \quad n > 3, \tag{1.50}$$

where

$$c_n = \frac{(2/\pi)^{\frac{1}{2}}}{2^{\frac{n-1}{2}}\,\Gamma(\frac{n-1}{2})}\ , \tag{1.51}$$

and

$$J_n = \int_0^{\infty} r^{n-1}\, e^{-\frac{1}{2}[(1+\alpha^2)r^2 - 2\alpha hr + h^2]}\, dr\ . \tag{1.52}$$

The recursion relation

$$J_n = \frac{\alpha h}{1+\alpha^2} J_{n-1} + \frac{n-2}{1+\alpha^2} J_{n-2}\ , \ n > 2\ , \tag{1.53}$$

may be easily obtained by integration by parts. For $G_n = c_{n+2} J_n \csc \omega$, $b_n = \dfrac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})}$, and $\delta = \frac{h}{2}$, a set of three recursion formulas

$$Q_n(\omega) = Q_{n-2}(\omega) + \cos \omega \, G_{n-2} \, , \; n > 3 \, , \tag{1.54a}$$

$$G_n = \cos \omega \sin \omega \, b_n G_{n-1} + \frac{n-2}{n-1} \sin^2 \omega \, G_{n-2} \, , \; n > 2 \, , \tag{1.54b}$$

$$b_n = \frac{n-2}{n-1} b_{n-2} \, , \; n > 2 \, , \tag{1.54c}$$

with initial values $b_1 = (\pi)^{\frac{1}{2}}$, $b_2 = 2(\pi)^{-\frac{1}{2}}$, $G_1 = \frac{1}{2} e^{-\delta^2} \sin^2 \omega \, \mathrm{erfc}(\delta)$, $G_2 = \frac{1}{\pi} \sin \omega \, e^{-\delta^2} + \frac{2\delta}{(\pi)^{\frac{1}{2}}} \sin \omega \cos \omega \, G_1$, $Q_3(\omega) = \frac{1}{2} \mathrm{erfc}(\delta) + \cos \omega \, G_1$

where

$$\mathrm{erfc}(x) = \frac{2}{(\pi)^{\frac{1}{2}}} \int_x^{\infty} e^{-y^2} dy \, , \tag{1.55}$$

will represent a tedious but elementary algorithm for evaluation of $Q_n(\omega)$. For the upper bound Slepian (13) obtains

$$Q(\omega_1) = \frac{M}{(\pi)^{\frac{1}{2}} b_{n-1}} \int_0^{\omega_1} Q_n(\omega) \sin^{n-2} \omega \, d\omega \, . \tag{1.56}$$

It is convenient to restrict these expressions to odd values of the dimensionality parameter n.

Numerical values of $Q_n(\omega)$ for given values of h are obtained rather easily from the above recursion formulas. For $n = 101$, which Slepian (13) takes as a reasonable upper value, 49 applications are required. To determine numerical values of $\bar{Q}(\omega_1)$ a trapezoidal method of integration utilizing 150 points is used. For $n = 101$ over 7000 applications of the recursion formulas are required to obtain a value of the upper bound. It is at this point that the alternate methods to be presented should have their greatest value.

EXTENSION OF THE SHANNON-SLEPIAN RESULTS

To obtain different results from those of Shannon (12) and Slepian (13) let us return to (1.32). Let $Q_n(\omega)$ for any odd integer n represent the lower bound of the probability of error in decoding, so $\Gamma(\frac{n-1}{2}) = (\frac{n-3}{2})!$, $n \geq 3$. By substituting $z = y + h$ and reversing the order of integration--certainly permissible for such an integrand--we obtain

$$Q_n(\omega) = \frac{1}{(2\pi)^{\frac{1}{2}} 2^{\frac{n-3}{2}} (\frac{n-3}{2})!} \left[\int_0^{\infty} e^{-(z-h)^2/2} dz \int_{\frac{z}{\alpha}}^{\infty} x^{n-2} e^{-x^2/2} dx \right. \tag{2.1}$$

$$\left. + \int_{-\infty}^{0} e^{-(z-h)^2/2} dz \int_0^{\infty} x^{n-2} e^{-x^2/2} dx \right] .$$

For the first special case,

$$Q_3(\omega) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_0^{\infty} e^{-(z-h)^2/2 - z^2/2\alpha^2} dz + \frac{1}{(2\pi)^{\frac{1}{2}}} \int_0^{\infty} e^{-(z+h)^2/2} dz$$

$$= \cos\omega\, \varphi^{(0)}(h \sin\omega) \int_0^{\infty} e^{-(u-h\cos\omega)^2/2} du + Q_1(h) , \tag{2.2}$$

where $\alpha = \cot\omega$, the variable of integration has been changed by $z = u \cos\omega$, and

$$Q_1(h) = \frac{1}{2} - \varphi^{(-1)}(h) . \tag{2.3}$$

The Gaussian density function and this form of the error function,

$$\begin{aligned} \varphi^{(0)}(x) &= \frac{1}{(2\pi)^{\frac{1}{2}}} e^{-x^2/2} , \\ \varphi^{(-1)}(x) &= \int_0^x \varphi^{(0)}(y)\, dy, \quad -\infty < x < \infty, \end{aligned} \tag{2.4}$$

were well-tabulated by the Computation Laboratory at Harvard University (4). Note that $Q_1(h)$ is the probability that a message point will be moved beyond the h point on a line, namely

$$Q_1(h) = \int_h^\infty \varphi^{(0)}(x)\, dx . \tag{2.5}$$

For $n = 5$,

$$\begin{aligned} Q_5(\omega) &= \frac{1}{2(2\pi)^{\frac{1}{2}}} \int_0^\infty e^{-(z-h)^2/2} dz \int_{\frac{z}{\alpha}}^\infty x^3 e^{-x^2/2} dx \\ &\quad + \frac{1}{2(2\pi)^{\frac{1}{2}}} \int_{-\infty}^0 e^{-(z-h)^2/2} dz \int_0^\infty x^3 e^{-x^2/2} dx \end{aligned} \tag{2.6}$$

$$= \frac{1}{2(2\pi)^{\frac{1}{2}}} \int_0^\infty e^{-(z-h)^2/2 - z^2/2\alpha^2} (\frac{z^2}{\alpha^2} + 2) dz + \frac{1}{(2\pi)^{\frac{1}{2}}} \int_0^\infty e^{-(z+h)^2/2} dz$$

$$= \frac{\sin^2 \omega}{2} \cos\omega\ \varphi^{(0)}(h \sin\omega) \int_0^\infty e^{-(u - h\cos\omega)^2/2} u^2 du + Q_3(\omega) .$$

A general recursion formula may be developed by integrating by parts; (2.1) may be written in the form (for n odd, ≥ 3)

$$Q_n(\omega) = \frac{1}{(2\pi)^{\frac{1}{2}} 2^{\frac{n-3}{2}} (\frac{n-3}{2})!} \left[\int_0^\infty e^{-(z-h)^2/2} dz \int_{\frac{z}{\alpha}}^\infty x^{n-2} e^{-x^2/2} dx \right. \tag{2.7}$$

$$\left. + \int_0^\infty e^{-(z+h)^2/2} dz \int_0^\infty x^{n-2} e^{-x^2/2} dx \right]$$

$$= \frac{1}{(2\pi)^{\frac{1}{2}} 2^{\frac{n-3}{2}} (\frac{n-3}{2})!} \int_0^\infty e^{-(z-h)^2/2} dz \left[(\tfrac{z}{\alpha})^{n-3} e^{-z^2/2\alpha^2} + (n-3) \int_{\frac{z}{\alpha}}^\infty x^{n-4} e^{-x^2/2} dx \right]$$

$$+ \frac{1}{(2\pi)^{\frac{1}{2}}} \int_0^\infty e^{-(z+h)^2/2} dz$$

$$= \left(\frac{\sin^2\omega}{2}\right)^{\frac{n-3}{2}} \frac{\cos\omega \ \varphi^{(0)}(h \sin\omega)}{(\frac{n-3}{2})!} \int_0^\infty e^{-(u-h\cos\omega)^2/2} u^{n-3} du$$

$$+ \frac{1}{(2\pi)^{\frac{1}{2}} 2^{\frac{n-5}{2}} (\frac{n-5}{2})!} \int_0^\infty e^{-(z-h)^2/2} dz \int_{\frac{z}{\alpha}}^\infty x^{n-4} e^{-x^2/2} dx + \frac{1}{(2\pi)^{\frac{1}{2}}} \int_0^\infty e^{-(z+h)^2/2} dz$$

$$= Q_{n-2}(\omega) + \left(\frac{\sin^2\omega}{2}\right)^{\frac{n-3}{2}} \frac{\cos\omega \ \varphi^{(0)}(h \sin\omega)}{(\frac{n-3}{2})!} \int_0^\infty e^{-(u-h\cos\omega)^2/2} u^{n-3} du ,$$

$$n \geq 3 .$$

It is convenient to change the index,

$$r = \frac{n-1}{2} , \tag{2.8}$$

so r takes values 0, 1, 2,... , as n ranges over the odd integers. Let $Q_n(\omega)$ be replaced by $P_r(\omega)$, with, in particular, $P_o(\omega) = Q_1(h)$.

Equation (2.7) now takes the form

$$P_r(\omega) = P_{r-1}(\omega) + \left(\frac{\sin^2\omega}{2}\right)^{r-1} \frac{\cos\omega\, \varphi^{(0)}(h\sin\omega)}{(r-1)!} \int_0^\infty e^{-(u - h\cos\omega)^2/2} u^{2r-2} du \,, \tag{2.9}$$

$$r = 1, 2, \ldots \,.$$

By repetitively applying this recursion formula we obtain, finally,

$$P_r(\omega) = P_o(h) + \cos\omega\, \varphi^{(0)}(h\sin\omega) \int_0^\infty e^{-(u - h\cos\omega)^2/2} \sum_{k=o}^{r-1} \frac{u^{2k}\sin^{2k}\omega}{2^k k!} du \,, \tag{2.10}$$

$$r \geq 0 \,.$$

For relatively small values of r a closed expression would be practicable. For larger values of r we are reminded of the cumulative Poisson distribution, tabulated by Molina (8). He writes

$$P(c,a) = \sum_{k=c}^{\infty} \frac{a^k e^{-a}}{k!} \,; \tag{2.11}$$

so

$$e^{-u^2\sin^2\omega/2} \sum_{k=o}^{r-1} \frac{u^{2k}\sin^{2k}\omega}{2^k k!} = 1 - P\left(r, \frac{u^2\sin^2\omega}{2}\right) = F\left(r, \frac{u^2\sin^2\omega}{2}\right) \,, \tag{2.12}$$

and the lower bound of the probability of error in decoding takes the form

$$P_r(\omega) = P_o(h) + \cos\omega \int_0^\infty \varphi^{(0)}(u\cos\omega - h)\, F\left(r, \frac{u^2\sin^2\omega}{2}\right) du \,. \tag{2.13}$$

Clearly this integral is dominated by the exponential factor although the off-center term in the exponent rules out the several elegant procedures which are available for numerical evaluation of integrals involving exponential factors. A simple trapezoidal or Simpson procedure might well be indicated.

An alternative procedure is to evaluate the integration in (2.10), leaving a finite sum. This may be accomplished in terms of the Hh-function, defined as

$$Hh_n(x) = \int_0^\infty \frac{t^n}{n!} e^{-(t+x)^2/2} dt, \quad -\infty < x < \infty, \tag{2.14}$$

and extensively tabulated by the British Association for the Advancement of Science (2). Thus (2.10) may be written in the form

$$P_r(\omega) = P_o + \cos\omega\, \varphi^{(0)}(h \sin\omega) \sum_{k=0}^{r-1} \frac{(2k)!}{2^k k!} \sin^{2k}\omega\, Hh_{2k}(-h\cos\omega). \tag{2.15}$$

An exact closed expression for the lower bound of the probability of error in decoding is attainable. It is convenient to introduce a slight variation of the Hh-function, say

$$I_n(x) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_0^\infty \frac{t^n}{n!} e^{-(t-x)^2/2} dt, \qquad n \geq 0, \tag{2.16}$$

and to define a set of Hermite polynomials with positive coefficients,

$$H_n(x) = e^{-x^2/2} \frac{d^n}{dx^n} e^{x^2/2}. \tag{2.17}$$

The first seven such polynomials are

$$H_0(x) = 1\,, \qquad H_4(x) = x^4 + 6x^2 + 3$$
$$H_1(x) = x\,, \qquad H_5(x) = x^5 + 10x^3 + 15x\,, \tag{2.18}$$
$$H_2(x) = x^2 + 1, \qquad H_6(x) = x^6 + 15x^4 + 45x^2 + 15,$$
$$H_3(x) = x^3 + 3x\,, \qquad \cdots\cdots\cdots\cdots\cdots .$$

and, in general,

$$H_n(x) = n! \sum_{k=o}^{[\frac{n}{2}]} \frac{x^{n-2k}}{2^k k!\,(n-2k)!} \,. \tag{2.19}$$

The recursion formula is at once

$$H_{n+1}(x) = xH_n(x) + nH_{n-1}(x)\,, \qquad n \geq 0\,. \tag{2.20}$$

Integration by parts in (2.16) yields the recursion formula

$$(n+1)I_{n+1}(x) = xI_n(x) + I_{n-1}(x)\,, \quad n \geq 1, \tag{2.21}$$

initial values $I_o(x) = \frac{1}{2} + \varphi^{(-1)}(x)$ and $I_1(x) = \varphi^{(o)}(x) + xI_o(x)$.

A few special cases are instructive and a conjecture is rather easily arrived at, namely

$$n!\,I_n(x) = H_n(x)\,I_o(x) + \varphi^{(0)}(x) \sum_{k=o}^{[\frac{n}{2}]} \frac{(n-k-1)!}{(n-2k-1)!} H_{n-2k-1}(x) \tag{2.22}$$

$$= H_n(x)\,I_o(x) + \varphi^{(0)}(x)\Big[H_{n-1}(x) + (n-2)H_{n-3}(x) + (n-3)(n-4)H_{n-5}(x) + (n-4)(n-5)(n-6)H_{n-7}(x) + \cdots\cdots + \begin{matrix} (\frac{n}{2})!\,H_1(x)\,, & n \text{ even}, \\ (\frac{n-1}{2})!\,H_o(x)\,, & n \text{ odd} \end{matrix} \Big]\,.$$

It is sufficient to show that the two easily established recursion formulas, (2.20) and (2.21), are consistent with this conjecture. That is, by rearrangement of terms,

$$(n+1)I_{n+1} - x I_n - I_{n-1} = \frac{1}{n!}[H_{n+1} - x H_n - n H_{n-1}]\, I_o(x)$$

$$+ \frac{1}{n!}\varphi^{(0)}(x) \Big[H_n - x H_{n-1} - (n-1)H_{n-2} + (n-2)[H_{n-2} - x H_{n-3} - (n-3)H_{n-4}] \tag{2.23}$$

$$+ (n-3)(n-4)[H_{n-4} - x H_{n-5} - (n-5)H_{n-6}] + \ldots\ldots\ldots \Big] ,$$

and, of course, each successive group of three terms on the right is identically zero. So, (2.10) may be written as

$$P_r(\omega) = P_o(h) + (2\pi)^{\frac{1}{2}} \cos\omega\, \varphi^{(0)}(h \sin\omega) \sum_{k=o}^{r-1} \frac{(2k)!}{2^k k!} \sin^{2k}\omega\, I_{2k}(h \cos\omega)$$

$$= P_o(h) + (2\pi)^{\frac{1}{2}} \cos\omega\, \varphi^{(0)}(h \sin\omega) \sum_{k=o}^{r-1} \frac{\sin^{2k}\omega}{2^k k!}[H_{2k}(h\cos\omega) I_o(h\cos\omega) \tag{2.24}$$

$$+ \varphi^{(0)}(h\cos\omega) \sum_{j=o}^{k-1} \frac{(2k-j-1)!}{(2k-2j-1)!} H_{2k-2j-1}(h\cos\omega)\ ] .$$

This exact solution is relatively easy to evaluate for small values of r, and there is no restriction on the value of $h \cos\omega$, which involves signal to noise ratio and the number of signal points scattered on the spherical surface in n-dimensional space.

Shannon (12) points out that the lower bound of the probability of error in decoding is equivalent to the non-central t-distribution,

$$Q(\omega) = F[n-1,\ (nA)^{\frac{1}{2}},\ (n-1)^{\frac{1}{2}} \cot\omega] , \tag{2.25}$$

which has received much study recently. It was tabulated to some extent by Johnson and Welch (7).

Let us now express our upper bound on the probability of decoding error, $\bar{Q}_n(\omega_1)$, using our expressions for the lower bound. From (1.34)

$$P_e \leq \bar{Q}_n(\omega_1) = \frac{M\Gamma(\frac{n}{2})}{(\pi)^{\frac{1}{2}}\Gamma(\frac{n-1}{2})} \int_0^{\omega_1} Q_n(\omega) \sin^{n-2} \omega \, d\omega \,, \qquad (2.26)$$

which may be written in the form

$$\begin{aligned} \bar{P}_r(\omega_1) &= \frac{M\Gamma(r+\frac{1}{2})}{(\pi)^{\frac{1}{2}}(r-1)!} \int P_r(\omega) \sin^{2r-1} \omega \, d\omega \\ &= \frac{M\,(2r)!}{2^{2r}\, r!\,(r-1)!} \int_0^{\omega_1} P_r(\omega) \sin^{2r-1} \omega \, d\omega \,. \end{aligned} \qquad (2.27)$$

The Poisson method of (2.13) then yields

$$\begin{aligned} \bar{P}_r(\omega_1) = {} & \frac{M\,(2r)!}{2^{2r}\, r!\,(r-1)!} \Big[\omega_1 P_o(h) \\ & + \int_0^{\omega_1} \cos\omega \sin^{2r-1} \omega \, d\omega \int_0^{\infty} \varphi^{(0)}(u \cos\omega - h)\, F(r, \frac{u^2 \sin^2 \omega}{2}) du \Big] \,. \end{aligned} \qquad (2.28)$$

Several numerical techniques are available for such an integration over a semi-infinite strip. Certainly the various integrand factors are very well-behaved. Molina's (6) Table II of P(c,a) has the range $c = 0(1)153$ while $a = .001(.001).01(.01).30(.1)15(1)100$, 6D.

The Hh method of (2.15) yields

$$\bar{P}_r(\omega_1) = \frac{M\,(2r)!}{2^{2r}\,r!\,(r-1)!}\Big[\,\omega_1 P_o(h) \tag{2.29}$$

$$+ \sum_{k=o}^{r-1} \frac{(2k)!}{2^k k!} \int_0^{\omega_1} \cos\omega \sin^{2r+2k-1}\omega\ \varphi^{(0)}(h\sin\omega)\ Hh_{2k}(-h\cos\omega)\,d\omega\,\Big].$$

For relatively small values of $h\cos\omega$ and r this may also be evaluated numerically. The Hh tables have the range $x = -7(.1)6.6$, $n = -7(1)21$, 10D.

Finally, the exact expression for upper bound on the probability of error in decoding takes the form

$$\bar{P}_r(\omega_1) = \frac{M\,(2r)!}{2^{2r}\,r!\,(r-1)!}\Big[\,\omega_1 P_o(h) \tag{2.30}$$

$$+ (2\pi)^{\frac{1}{2}} \sum_{k=o}^{r-1} \frac{(2k)!}{2^k k!} \int_0^{\omega_1} \cos\omega \sin^{2r+2k-1}\omega\ \varphi^{(0)}(h\sin\omega)\ I_{2k}(h\cos\omega)\,d\omega\,\Big]$$

$$= \frac{M\,(2r)!}{2^{2k}\,r!\,(r-1)!}\Bigg[\omega_1 P_o + (2\pi)^{\frac{1}{2}} \sum_{k=o}^{r-1} \frac{1}{2^k k!} \int_0^{\omega_1} \cos\omega \sin^{2r+2k-1}\omega\ \varphi^{(0)}(h\sin\omega)$$

$$\cdot\Big[\,H_{2k}(h\cos\omega)\,I_o(h\cos\omega)$$

$$+ \varphi^{(0)}(h\cos\omega) \sum_{j=o}^{k-1} \frac{(2k-j-1)!}{(2k-2j-1)!} H_{2k-2j-1}(h\cos\omega)\,d\omega\Big]\Bigg].$$

Note that the product of the two Gaussian density functions is a mere constant. The presence of the $\cos\omega$ factor in the integrand implies that a change of variable of integration might be worthwhile. Set $x = \sin\omega$ and

$$\bar{P}_r(\omega_1) = \frac{M\,(2r)!}{2^{2r} r!(r-1)!}\left[\omega_1 P_o + (2\pi)^{\frac{1}{2}} \sum_{k=o}^{r-1} \frac{1}{2^k k!} \int_0^{\sin\omega_1} x^{2r+2k-1} \varphi^{(0)}(hx)\right. \tag{2.31}$$

$$\cdot\left[H_{2k}(h\sqrt{1-x^2})\, I_o(h\sqrt{1-x^2})\right.$$

$$\left.\left. + \varphi^{(0)}(h\sqrt{1-x^2}) \sum_{j=o}^{k-1} \frac{(2k-j-1)!}{(2k-2j-1)!} H_{2k-2j-1}(h\sqrt{1-x^2})\right]dx\right] .$$

Still another form of this would result for $x = \cos\omega$:

$$\bar{P}_r(\omega_1) = \frac{M\,(2r)!}{2^{2r} r!(r-1)!}\left[\omega_1 P_o + (2\pi)^{\frac{1}{2}} \sum_{k=o}^{r-1} \frac{1}{2^k k!} \int_{\cos\omega_1}^{1} x(1-x^2)^{r+k-1} \varphi^{(0)}(h\sqrt{1-x^2})\right. \tag{2.32}$$

$$\left.\cdot\left[H_{2k}(hx)\, I_o(hx) + \varphi^{(0)}(hx) \sum_{j=o}^{k-1} \frac{(2k-j-1)!}{(2k-2j-1)!} H_{2k-2j-1}(hx)\ \right] dx\right] .$$

The evaluation of the several definite integrals developed above should offer no great difficulty as all functions involved are very well-behaved. We are assured that trapezoidal rule methods of integrating apply in each case as the integrand functions have continuous derivatives of all orders (6, p.256-259). A paper by Burgoyne (3) is also useful in this context.

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