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Title PROOFS AND COMPARISONS OF THE MANN-DYSON ESTIMATION THEOREMS

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Let A and B be two subsets of the set of all non-negative integers with $0 \in A$ and $0 \in B$. The sum of the sets A and B is the set $C = A + B = \{a + b\colon a \in A,\ b \in B\}$. For $n > 0$ we denote the number of positive integers in C which do not exceed n by $C(n)$.

In this thesis we obtain and compare four lower bounds for $C(n)$. These lower bounds are consequences of well known theorems from additive number theory. The word estimate is frequently used for such lower bounds which explains the title of this thesis.

PROOFS AND COMPARISONS OF THE MANN-DYSON ESTIMATION THEOREMS

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PROOFS AND COMPARISONS OF THE MANN-DYSON ESTIMATION THEOREMS

CHAPTER I

Introduction

Let A and B be two subsets of the set of all non-negative integers with $0 \in A$ and $0 \in B$. The sum of the sets A and B is the set $C = A + B = \{a + b: a \in A, b \in B\}$. For $n > 0$ we denote the number of positive integers in A which do not exceed n by $A(n)$. We define $B(n)$ and $C(n)$ similarly.

In 1931 L. G. Schnirelmann (9) introduced the Schnirelmann density of such a set A. Letting α denote the Schnirelmann density of the set A we have

$$\alpha = \underset{n>0}{\text{g.}\ell\text{.b.}} \frac{A(n)}{n}.$$

Since the formulation of the Schnirelmann density, many inequalities involving the Schnirelmann density of the sets A, B and C have been proved. Among these is an inequality whose proof was attempted with little success by many mathematicians. Letting α, β and γ denote the Schnirelmann densities of A, B and C respectively it was conjectured that $\gamma = 1$ or $\gamma \geq \alpha + \beta$.

Finally in 1942 H. B. Mann (5) obtained the result $C(n) = n$, or both $C(n) < n$ and

$$\frac{C(n)}{n} \geq \min_{\substack{1 \leq m \leq n \\ m \notin C}} \frac{A(m) + B(m)}{m} .$$

This result implies the conjecture which has since become commonly known as the $\alpha\beta$ Theorem of additive number theory. However, the interest here is not in the $\alpha\beta$ Theorem as such but rather in the lower bound for $C(n)$ obtainable from the above inequality of Mann.

The above inequality of Mann is explicitly or implicitly contained in later works by Emil Artin and Peter Scherk (1), A. Y. Khinchin (4), and Leroy Mitchell Damewood (2).

In this thesis we obtain and compare four lower bounds for $C(n)$. These lower bounds are consequenees of well known theorems from additive number theory. The word estimate is frequently used for such lower bounds which explains the title of this thesis. We display the four lower bounds for $C(n)$ in the form of inequalities in the following theorem whose proof is delayed until the next chapter.

Theorem 1.1. Let $C = A + B$ with $0 \in A$ and $0 \in B$. If $n > 0$ then $C(n) = n$, or both $C(n) < n$ and

(a) $$C(n) \geq \min_{1 \leq m \leq n} \frac{A(m) + B(m)}{m} n .$$

(b) $$C(n) \geq \min_{\substack{1 \leq m \leq n \\ m \notin C}} \frac{A(m) + B(m)}{m} n.$$

(c) $$C(n) \geq \min_{1 \leq m \leq n} \frac{A(m) + B(m) + 1}{m+1} (n+1) - 1.$$

(d) $$C(n) \geq \min_{\substack{1 \leq m \leq n \\ m \notin C}} \frac{A(m) + B(m) + 1}{m+1} (n+1) - 1.$$

As we point out in Chapter II, part (b) is an immediate consequence of the inequality of Mann (5). Also we show how parts (a) and (c) are obtained from two results from a paper by J. G. Van der Corput (10).

Van der Corput credits F. J. Dyson (3) with obtaining part (a) in a more general form. Van der Corput uses a modification of the Dyson method to obtain both of his results. Since parts (a) and (c) are consequences of these two results we refer to parts (a) and (c) as Dyson's estimates of $C(n)$. Dyson's result in a weaker form also occurs in a book by Hans-Heinrich Ostmann (8) and in a book authored jointly by I. Niven and H. S. Zuckerman (7), and part (a) is obtainable from these works also.

Also we point out in Chapter II that part (d) is an immediate consequence of a result which appeared in a later paper by Mann (6). His proof of this result is very condensed and difficult to read.

In Chapter II we establish the four estimates of $C(n)$ displayed in Theorem 1. 1 as consequences of the above mentioned results and in Chapter III we compare these estimates.

Since $C(n)$ is an integer, perhaps of greater interest than the lower bounds for $C(n)$ are the smallest integers greater than or equal to these lower bounds. With this in mind we establish in Chapter IV a theorem analogous to Theorem 1. 1, which we recall yields our estimates of $C(n)$, but with these estimates replaced by the smallest integers greater than or equal to these estimates. We then compare these new estimates also.

In Chapter V and VII we rewrite with much greater detail the proof of the result of Mann (6) which gives part (d) of Theorem 1. 1. In Chapter VI we give a variation of part of Mann's proof.

CHAPTER II

Proofs of the Mann-Dyson Estimation Theorems

In preparation for the proof of Theorem 1. 1 we establish a well known lemma which will be used frequently throughout this thesis. Also, we list without proof a theorem of Van der Corput (10) and a theorem of Mann (6).

Lemma 2. 1. Let $C = A + B$ with $0 \in A$ and $0 \in B$. If $m > 0$ and $m \notin C$ then $A(m) + B(m) + 1 \leq m$.

Proof: For each $a \in A$ such that $0 \leq a \leq m$ we have $(m - a) \notin B$, for if we assume $(m - a) \in B$ for some $a \in A$ then $a + (m - a) = m \in C$ contrary to hypothesis. Also $0 < m - a \leq m$. There are $A(m) + 1$ such integers not in B. Thus we have $B(m) \leq m - [A(m) + 1]$, or $A(m) + B(m) + 1 \leq m$.

Theorem 2. 2. (Van der Corput) Let $C = A + B$ with $0 \in A$ and $0 \in B$.

(a) If

$A(m) + B(m) \geq \gamma m$ for $m = 1, 2, \ldots, n$, where $\gamma \leq 1$ and n is a positive integer, then

$C(m) \geq \gamma m$ for $m = 1, 2, \ldots, n$;

(b) If

$A(m) + B(m) + 1 \geq \gamma(m + 1)$ for $m = 1, 2, \ldots, n$, where

$\gamma \leq 1$ and n is a positive integer, then

$C(m) + 1 \geq \gamma(m + 1)$ for $m = 1, 2, \ldots, n$.

Theorem 2.3 (Mann) Let $C = A + B$ with $0 \in A$ and $0 \in B$. For $n > 0$ either $C(n) = n$, or $C(n) < n$ and there exist numbers m and m_1 such that $m \notin C$, $0 < m \leq n$, $m_1 \notin C$, and $0 < m_1 \leq \max(m, n - m - 1)$, for which

$$\frac{C(n)+1}{n+1} \geq \frac{A(m) + B(m) + 1}{m+1} + \left| \frac{C(n)+1}{n+1} - \frac{C(m_1)+1}{m_1+1} \right| .$$

For convenience we introduce some additional notation. Let f and g be functions of the integer m such that $f(m) = \frac{A(m) + B(m)}{m}$ and $g(m) = \frac{A(m) + B(m) + 1}{m+1}$. Also, define the sets of integers S and S^* by $S = \{m: 1 \leq m \leq n\}$ and $S^* = \{m: 1 \leq m \leq n, m \notin C\}$. Clearly S^* is a subset of S.

We now restate Theorem 1.1 in the following form.

Theorem 2.4 Let $C = A + B$ with $0 \in A$ and $0 \in B$. If $n > 0$ then $C(n) = n$, or both $C(n) < n$ and

(a) $C(n) \geq \min\limits_{m \in S} n f(m)$

(b) $C(n) \geq \min\limits_{m \in S^*} n f(m)$

(c) $C(n) \geq \min\limits_{m \in S} (n+1) g(m) - 1$

(d) $C(n) \geq \min_{m \in S^*} (n+1)g(m) - 1$.

Proof. Part (b) is merely the result of isolating $C(n)$ in the inequality of Mann (5). Thus

$$C(n) \geq \min_{\substack{1 \leq m \leq n \\ m \notin C}} n \frac{A(m) + B(m)}{m} = \min_{m \in S^*} nf(m).$$

From Theorem 2.3 we have that $C(n) = n$, or $C(n) < n$ and there exists an m such that $m \notin C$, $1 \leq m \leq n$, and $\frac{C(n)+1}{n+1} \geq \frac{A(m) + B(m) + 1}{m+1}$. Thus,

$$C(n) \geq (n+1) \frac{A(m) + B(m) + 1}{m+1} -1 \geq \min_{\substack{1 \leq m \leq n \\ m \notin C}} (n+1) \frac{A(m) + B(m) + 1}{m+1} -1$$

$$= \min_{m \in S^*} (n+1)g(m) - 1,$$

which establishes part (d).

We make use of Theorem 2.2 to prove parts (a) and (c). First we prove part (a).

If $\min_{1 \leq m \leq n} \frac{A(m) + B(m)}{m} \leq 1$, then in Theorem 2.2 (a) we choose $\gamma = \min_{1 \leq m \leq n} \frac{A(m) + B(m)}{m}$ and have

$$\frac{C(n)}{n} \geq \min_{1 \leq m \leq n} \frac{A(m) + B(m)}{m}, \text{ so that}$$

$$C(n) \geq \min_{1 \leq m \leq n} n \frac{A(m) + B(m)}{m} = \min_{m \in S} n f(m).$$

If $\min_{1 \leq m \leq n} \frac{A(m) + B(m)}{m} > 1$ then $A(m) + B(m) > m > m - 1$ for all m, $1 \leq m \leq n$, so that $m \in C$ by Lemma 2.1 and $C(n) = n$.

We prove part (c) in a similar fashion.

If $\min_{1 \leq m \leq n} \frac{A(m) + B(m) + 1}{m+1} \leq 1$, then in Theorem 2.2 (b) we chose $\gamma = \min_{1 \leq m \leq n} \frac{A(m) + B(m) + 1}{m+1}$ and have

$$\frac{C(n)+1}{n+1} \geq \min_{1 \leq m \leq n} \frac{A(m) + B(m) + 1}{m+1}, \text{ so that}$$

$$C(n) \geq \min_{1 \leq m \leq n} (n+1) \frac{A(m) + B(m) + 1}{m+1} - 1 = \min_{m \in S} (n+1) g(m) - 1.$$

If $\min_{1 \leq m \leq n} \frac{A(m) + B(m) + 1}{m+1} > 1$ then $A(m) + B(m) + 1 > m + 1$ or $A(m) + B(m) > m > m - 1$ for all m, $1 \leq m \leq n$, so that $m \in C$ by Lemma 2.1 and $C(n) = n$.

CHAPTER III

Comparisons of the Mann-Dyson Estimation Theorems

Having obtained the four lower bounds for $C(n)$ of Theorem 2.4, we now compare them.

Theorem 3.1. Let $C = A + B$ with $0 \in A$ and $0 \in B$. If $n > 0$ and $C(n) < n$, then

(a) $\min\limits_{m \in S} n\, f(m) \leq \min\limits_{m \in S^*} n\, f(m)$;

(b) $\min\limits_{m \in S^*} n\, f(m) \leq \min\limits_{m \in S^*} (n+1)g(m) - 1$;

(c) $\min\limits_{m \in S} n\, f(m) \leq \min\limits_{m \in S} (n+1)g(m) - 1$;

(d) $\min\limits_{m \in S} (n+1)g(m) - 1 \leq \min\limits_{m \in S^*} (n+1)g(m) - 1$;

(e) there exist examples where $\min\limits_{m \in S^*} n\, f(m) < \min\limits_{m \in S}(n+1)g(m) - 1$,

$\min\limits_{m \in S^*} n\, f(m) = \min\limits_{m \in S} (n+1)g(m) - 1$, and

$\min\limits_{m \in S^*} n\, f(m) > \min\limits_{m \in S} (n+1)g(m) - 1$.

Proof of parts (a) and (d). Since the minimum of a subset is not less than the minimum of the set we have parts (a) and (d).

To establish parts (b) and (c) we obtain the following lemma.

Lemma 3.2. If $0 < m < n$ and $g(m) < 1$, then $(n+1)g(m) - 1 > n\,f(m)$.

Proof. We have

$$1 > g(m) = \frac{m\,f(m) + 1}{m+1},$$

and so $m\,f(m) + 1 < m + 1$, or $f(m) < 1$. Hence

$$n - m > (n - m)f(m).$$

Addition of $m(n+1)f(m)$ to each member yields

$$m(n+1)f(m) + n - m > n(m+1)f(m),$$

whence

$$(n+1)[m\,f(m) + 1] - (m+1) > n(m+1)f(m).$$

Finally, dividing by $m+1$ we obtain

$$\begin{aligned}(n+1)g(m) - 1 &= (n+1)\frac{m\,f(m) + 1}{m+1} - 1 \\ &> n\,f(m).\end{aligned}$$

Now we prove part (c). Let m_1 be any $m \in S$ for which $(n+1)g(m) - 1$ is minimized. Then by hypothesis and Theorem 2.4 (c), we have

$$\begin{aligned}n > C(n) &\geq \min_{m \in S} (n+1)g(m) - 1 \\ &= (n+1)g(m_1) - 1,\end{aligned}$$

so that $n+1 > (n+1)g(m_1)$, or $g(m_1) < 1$. If $m_1 = n$, then

$$(n+1)g(m_1) - 1 = A(n) + B(n) = n\, f(m_1),$$

and if $m_1 < n$ we have by Lemma 3.2 that

$$(n+1)g(m_1) - 1 > n\, f(m_1).$$

Hence in either case

$$\min_{m \in S} (n+1)g(m) - 1 = (n+1)g(m_1) - 1$$

$$\geq n\, f(m_1) \geq \min_{m \in S} n\, f(m),$$

and the proof of part (c) is complete.

We obtain a proof of part (b) by replacing S by S^* and Theorem 2.4 (c) by Theorem 2.4 (d) in the proof of part (c).

We prove part (e) by giving an example of each type whose existence is asserted.

Example 1. Let $A = \{0, 2, 8, 9, \cdots\}$ and $B = \{0, 4, 8, 9, \cdots\}$. Then $C = \{0, 2, 4, 6, 8, 9, \cdots\}$. Let $n = 7$. Then $\min_{m \in S} (n+1)g(m) - 1 = 8\, g(7) - 1 = 2$, while $\min_{m \in S^*} n\, f(m) = 7\, f(1) = 0$, so that $\min_{m \in S} (n+1)g(m) - 1 > \min_{m \in S^*} n\, f(m)$.

Example 2. Let $A = \{0, 1, 2, 11, 12, \cdots\}$ and $B = \{0, 4, 8, 11, 12, \cdots\}$. Then $C = \{0, 1, 2, 4, 5, 6, 8, 9, \cdots\}$. Let $n = 7$. Then $\min_{m \in S} (n+1)g(m) - 1 = 8\, g(7) - 1 = 3$, while $\min_{m \in S^*} n\, f(m) = 7\, f(7) = 3$, so that $\min_{m \in S} (n+1)g(m) - 1 = \min_{m \in S^*} n\, f(m)$.

Example 3. Let $A = \{0, 1, 2, 8, 10, 11, 23, 24, \cdots\}$ and $B = \{0, 3, 5, 8, 9, 13, 23, 24, \cdots\}$. Then $C = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, \cdots\}$. Let $n = 12$. Then $\min_{m \in S} (n+1)g(m) - 1 = 13\,g(7) - 1 = \frac{57}{8}$ and $\min_{m \in S^*} n\,f(m) = 12f(12) = 9$ so that $\min_{m \in S} (n+1)g(m) - 1 < \min_{m \in S^*} n\,f(m)$.

This completes the proof of Theorem 3.1.

Theorem 3.3. Necessary conditions for equality to hold in parts (a), (b), (c) and (d) of Theorem 3.1 are respectively as follows. In the case of parts (a), (c) and (d) the condition is also sufficient.

(a) $\min_{m \in S} f(m) = \min_{m \in S^*} f(m)$;

(b) $\min_{m \in S^*} f(m) = f(n)$;

(c) $\min_{m \in S} f(m) = f(n)$;

(d) $\min_{m \in S} g(m) = \min_{m \in S^*} g(m)$.

Proof. Parts (a) and (d) are immediate so we turn to parts (b) and (c).

Proof of part (c). If $\min_{m \in S} f(m) = f(n)$, then by Theorem 3.1 (c), we have

$$\min_{m \in S} (n+1)g(m) - 1 \geq \min_{m \in S} n\, f(m)$$

$$= n\, f(n) = A(n) + B(n)$$

$$= (n+1)g(n) - 1$$

$$\geq \min_{m \in S} (n+1)g(m) - 1.$$

Consequently, $\min_{m \in S} (n+1)g(m) - 1 = \min_{m \in S} n\, f(m)$.

Conversely, suppose $\min_{m \in S} (n+1)g(m) - 1 = \min_{m \in S} n\, f(m)$. Let m_1 by any $m \in S$ for which $g(m)$ is minimized. Now $g(m_1) < 1$, for assume $g(m_1) \geq 1$. Then for each $m \in S$ we have

$$1 \leq g(m_1) \leq g(m) = \frac{A(m) + B(m) + 1}{m+1},$$

and so $A(m) + B(m) + 1 \geq m + 1 > m$. Hence $m \in C$ by Lemma 2.1 and it follows that $C(n) = n$, contrary to the hypothesis of Theorem 3.1. Thus $g(m_1) < 1$. Now $m_1 = n$, for assume $m_1 < n$. Since $g(m_1) < 1$ then we have by Lemma 3.2 that

$$\min_{m \in S} (n+1)g(m) - 1 = (n+1)g(m_1) - 1$$

$$> n\, f(m_1) \geq \min_{m \in S} n\, f(m),$$

contrary to hypothesis. Thus $m_1 = n$, and

$$\min_{m \in S} n\, f(m) = \min_{m \in S} (n+1)g(m) - 1$$

$$= (n+1)g(m_1) - 1$$

$$= A(n) + B(n) = n\, f(n),$$

so that $\min_{m \in S} f(m) = f(n)$. This completes the proof of part (c).

Proof of part (b). This is similar to the necessity part of the proof of part (c). Suppose $\min_{m \in S^*} (n+1)g(m) - 1 = \min_{m \in S^*} n\,f(m)$. Let m_1 be any $m \in S^*$ for which $g(m)$ is minimized. Now $g(m_1) < 1$, for assume $g(m_1) \geq 1$ and let $m \in S^*$. We have

$$1 \leq g(m_1) \leq g(m) = \frac{A(m) + B(m) + 1}{m+1},$$

and so $A(m) + B(m) + 1 \geq m + 1 > m$. Hence $m \in C$ by Lemma 2.1, which is contrary to $m \in S^*$. Thus $g(m_1) < 1$. Now $m_1 = n$, for assume $m_1 < n$. Since $g(m_1) < 1$, then we have by Lemma 3.2 that

$$\begin{aligned} \min_{m \in S^*} (n+1)g(m) - 1 &= (n+1)g(m_1) - 1 \\ &> n\,f(m_1) \geq \min_{m \in S^*} n\,f(m), \end{aligned}$$

contrary to hypothesis. Thus $m_1 = n$, and

$$\begin{aligned} \min_{m \in S^*} n\,f(m) &= \min_{m \in S^*} (n+1)g(m) - 1 \\ &= (n+1)g(n) - 1 \\ &= A(n) + B(n) = n\,f(n), \end{aligned}$$

so that $\min_{m \in S^*} f(m) = f(n)$. This completes the proof of part (b) and Theorem 3.3.

We would like to have found that condition (b) is also sufficient for equality to hold in part (b) of Theorem 3.1 but this is not always so.

When $n \notin C$ condition (b) is a sufficient condition for equality to hold in part (b) of Theorem 3.1 by a similar argument to that used in showing that condition (c) is sufficient for equality to hold in part (c) of Theorem 3.1. That is, if $n \notin C$, then $n \in S^*$ by our definition of S^*. If $\min_{m \in S^*} f(m) = f(n)$, then by Theorem 3.1 (b), we have

$$\begin{aligned}\min_{m \in S^*} (n+1)g(m) - 1 &\geq \min_{m \in S^*} n\, f(m)\\ &= n\, f(n) = A(n) + B(n)\\ &= (n+1)g(n) - 1\\ &\geq \min_{m \in S^*} (n+1)g(m) - 1\end{aligned}$$

Consequently, $\min_{m \in S^*} (n+1)g(m) - 1 = \min_{m \in S^*} n\, f(m)$ when $n \notin C$ and $\min_{m \in S^*} f(m) = f(n)$.

We now show an example where $n \in C$ and condition (b) holds but equality does not hold in part (b) of Theorem 3.1.

Example 4. Let $A = \{0, 1, 4, 6, 8, 13, 14, \cdots\}$ and $B = \{0, 1, 4, 5, 7, 13, 14, \cdots\}$. Then $C = \{0, 1, 2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, \cdots\}$. Let $n = 12$. Then $S^* = \{3\}$, and so

$$\min_{m \in S^*} f(m) = f(3) = \frac{2}{3} \text{ and } f(n) = f(12) = \frac{8}{12} = \frac{2}{3}.$$

However,

$$\min_{m \in S^*} (n+1)g(m) - 1 = 13g(3) - 1 = 13(\tfrac{3}{4}) - 1 = \frac{35}{4},$$

while

$$\min_{m \in S^*} n\, f(m) = 12 f(3) = 12\left(\frac{2}{3}\right) = 8.$$

Thus condition (b) is not sufficient for equality to hold in part (b) of Theorem 3.1 when $n \in C$.

We also would like to note that although we have no necessary and sufficient condition for the equality $\min_{m \in S} (n+1)g(m) - 1 = \min_{m \in S^*} n f(m)$ to hold, a necessary and sufficient condition is readily found for the equality

$$(3.1) \qquad \min_{m \in S} n\, f(m) = \min_{m \in S^*} (n+1)g(m) - 1$$

to hold. A necessary and sufficient condition for equality (3.1) to hold is the condition

$$(3.2) \qquad \min_{m \in S} g(m) = \min_{m \in S^*} g(m) \quad \text{and} \quad \min_{m \in S} f(m) = f(n).$$

To prove this we see that if condition (3.2) holds then by Theorem 3.3 parts (c) and (d) equality holds in parts (c) and (d) of Theorem 3.1. Thus equality (3.1) holds.

Conversely, if equality (3.1) holds then equality must hold in parts (c) and (d) of Theorem 3.1. By Theorem 3.3 parts (c) and (d) both parts of condition (3.2) hold.

We now give a theorem guaranteeing examples for all eight cases of equality and strict inequality in parts (a) through (d) of Theorem 3.1.

Theorem 3.4. There exist examples where each of the eight cases of equality and strict inequality hold in the following inequalities:

(a) $\min_{m \in S} n\,f(m) \le \min_{m \in S^*} n\,f(m)$;

(b) $\min_{m \in S^*} n\,f(m) \le \min_{m \in S^*} (n+1)g(m) - 1$;

(c) $\min_{m \in S} n\,f(m) \le \min_{m \in S} (n+1)g(m) - 1$;

(d) $\min_{m \in S} (n+1)g(m) - 1 \le \min_{m \in S^*} (n+1)g(m) - 1$.

Proof. We consider the following example.

Example 5: Let $A = \{0, 1, 2, 4, 5, \ldots\}$ and $B = \{0, 4, 5, \ldots\}$. Then $C = \{0, 1, 2, 4, 5, \ldots\}$. Let $n = 3$. Then

$$\min_{m \in S} f(m) = \min_{m \in S^*} f(m) = f(n) = \frac{2}{3} \text{ and } \min_{m \in S} g(m) = \min_{m \in S^*} g(m) = \frac{3}{4}.$$

Since $\min_{m \in S} f(m) = \min_{m \in S^*} f(m)$, then by Theorem 3.3 (a) we have an example illustrating the equality portion of Theorem 3.4 (a).

Since $\min_{m \in S^*} n\,f(m) = 3\,(\frac{2}{3}) = 2$ and $\min_{m \in S^*} (n+1)g(m) - 1 = 4\,(\frac{3}{4}) - 1 = 2$, we have an example illustrating the equality portion of Theorem 3.4 (b).

Since $\min_{m \in S} f(m) = f(n)$, then by Theorem 3.3 (c) we have an example illustrating the equality portion of Theorem 3.4 (c).

Since $\min_{m \in S} g(m) = \min_{m \in S^*} g(m)$, then by Theorem 3.3 (d) we have an example illustrating the equality portion of Theorem 3.4 (d).

Now consider previously listed Example 3 with $n = 12$. Then

$\min_{m \in S^*} f(m) = f(n) = \frac{3}{4}$, $\min_{m \in S} f(m) = \frac{4}{7}$, $\min_{m \in S} g(m) = \frac{5}{8}$, and

$\min_{m \in S^*} g(m) = \frac{10}{13}$.

Since $\min_{m \in S} f(m) \neq \min_{m \in S^*} f(m)$, then by Theorem 3.3 (a) we have an example illustrating the strict inequality portion of Theorem 3.4 (a).

Since $\min_{m \in S} f(m) \neq f(n)$, then by Theorem 3.3 (c) we have an example illustrating the strict inequality portion of Theorem 3.4 (c).

Since $\min_{m \in S} g(m) \neq \min_{m \in S^*} g(m)$, then by Theorem 3.3 (d) we have an example illustrating the strict inequality portion of Theorem 3.4 (d).

Now consider previously listed Example 1 with $n = 7$. Then

$\min_{m \in S^*} f(m) = 0$ and $f(n) = \frac{2}{7}$.

Since $\min_{m \in S^*} f(m) \neq f(n)$, then by Theorem 3.3 (b) we have an example illustrating the strict inequality portion of Theorem 3.4 (b). This completes the proof of Theorem 3.4.

CHAPTER IV

Integral Estimates

We proceed now to establish several theorems analogous to Theorems 2.4, 3.1, 3.3 and 3.4 but with each estimate replaced by the smallest integer greater than or equal to the estimate. For convenience we introduce further notation and establish another lemma.

Lemma 4.1. Let a be any real number and let $\langle a \rangle$ denote the smallest integer greater than or equal to a. Then if b is also any real number and

$$a \geq b$$

then

$$\langle a \rangle \geq \langle b \rangle .$$

Proof: If $a = b$, then $\langle a \rangle = \langle b \rangle$ and the lemma is true. If $a > b$ then $\langle b \rangle$ can be at most equal to $\langle a \rangle$. Thus $\langle a \rangle \geq \langle b \rangle$ and the lemma is proved.

Theorem 4.2. Let $C = A + B$ with $0 \in A$ and $0 \in B$. If $n > 0$ then $C(n) = n$, or both $C(n) < n$ and

(a) $C(n) \geq \left\langle \min\limits_{m \in S} n\, f(m) \right\rangle$;

(b) $C(n) \geq \left\langle \min\limits_{m \in S^*} n\, f(m) \right\rangle$;

(c) $C(n) \geq \left\langle \min\limits_{m \in S} (n+1)\, g(m) - 1 \right\rangle$;

(d) $C(n) \geq \left\langle \min_{m \in S^*} (n+1)g(m) - 1 \right\rangle$.

Proof: Since $C(n)$ is an integer $\langle C(n) \rangle = C(n)$. Thus inequalities (a) through (d) of Theorem 4.2 are just the inequalities (a) through (d) of Theorem 2.4 with the members of the inequalities replaced by the smallest integers greater than or equal to the members. Since the inequalities of Theorem 2.4 hold, by Lemma 4.1 the inequalities of Theorem 4.2 must also hold and the theorem is proved.

Theorem 4.3. Let $C = A + B$ with $0 \in A$ and $0 \in B$. If $n > 0$ and $C(n) < n$, then

(a) $\left\langle \min_{m \in S} n\, f(m) \right\rangle \leq \left\langle \min_{m \in S^*} n\, f(m) \right\rangle$;

(b) $\left\langle \min_{m \in S^*} n\, f(m) \right\rangle \leq \left\langle \min_{m \in S^*} (n+1)g(m) - 1 \right\rangle$;

(c) $\left\langle \min_{m \in S} n\, f(m) \right\rangle \leq \left\langle \min_{m \in S} (n+1)g(m) - 1 \right\rangle$;

(d) $\left\langle \min_{m \in S} (n+1)g(m) - 1 \right\rangle \leq \left\langle \min_{m \in S^*} (n+1)g(m) - 1 \right\rangle$;

(e) there exist examples where

$$\left\langle \min_{m \in S^*} n\, f(m) \right\rangle < \left\langle \min_{m \in S} (n+1)g(m) - 1 \right\rangle,$$

$$\left\langle \min_{m \in S^*} n\, f(m) \right\rangle = \left\langle \min_{m \in S} (n+1)g(m) - 1 \right\rangle, \text{ and}$$

$$\left\langle \min_{m \in S^*} n\, f(m) \right\rangle > \left\langle \min_{m \in S} (n+1)g(m) - 1 \right\rangle.$$

Proof. We prove inequalities (a) through (d) first. Since inequalities (a) through (d) of Theorem 4.3 are just the inequalities of Theorem 3.1 parts (a) through (d) with the members replaced by the smallest integers greater than or equal to the members, by Lemma 4.1 inequalities (a) through (d) of Theorem 4.3 also hold.

To prove part (e) of Theorem 4.3 we need only note that the examples used in the proof of Theorem 3.1 (e) also suffice for the proof of part (e) of Theorem 4.3. This completes the proof of Theorem 4.3.

We now prove one more lemma in preparation for further theorems.

Lemma 4.4. If a and b are any non-negative real numbers, m is any integer, and n is any positive integer such that

$$\langle na + m \rangle = \langle nb + m \rangle = K,$$

then

$$\langle a \rangle = \langle b \rangle .$$

Proof. Let $K' = K - 1$. Then since $na + m = nb + m = K' + 1$, we have $na + m = K' + \delta_1$, where $0 < \delta_1 \leq 1$, and $nb + n = K' + \delta_2$, where $0 < \delta_2 \leq 1$. Let $K'' = K' - m$. Then $na = K'' + \delta_1$ and $nb = K'' + \delta_2$. We may write $K'' = kn + r$ where k and r are integers and $0 \leq r < n$. Then $na = kn + r + \delta_1$ and $nb = kn + r + \delta_2$. Hence $a = k + \dfrac{r + \delta_1}{n}$ and $b = k + \dfrac{r + \delta_2}{n}$. Since $0 < \delta_1 \leq 1$ and $0 \leq r < n$, then $0 < r < \delta_1 \leq n$, and so $0 < \dfrac{r + \delta_1}{n} \leq 1$. Similarly $0 < \dfrac{r + \delta_2}{n} \leq 1$. Finally,

$a = k + 1 = b$, and the proof is complete.

Theorem 4.5. Necessary conditions for equality to hold in parts (a) and (d) of Theorem 4.3 and a sufficient condition for equality to hold in part (c) of Theorem 4.3 are respectively:

(a) $\left\langle \min_{m \in S} f(m) \right\rangle = \left\langle \min_{m \in S^*} f(m) \right\rangle$;

(b) $\left\langle \min_{m \in S} g(m) \right\rangle = \left\langle \min_{m \in S^*} g(m) \right\rangle$;

(c) $\left\langle \min_{m \in S} n \; f(m) \right\rangle = \left\langle n \, f(n) \right\rangle$.

Proof. We prove parts (a) and (b) first. If equality holds in parts (a) and (d) of Theorem 4.3 then by Lemma 4.4 equality also holds in parts (a) and (b) of Theorem 4.5.

We prove part (c) next. If $\left\langle \min_{m \in S} n \; f(m) \right\rangle = \left\langle n \, f(n) \right\rangle$, then by Theorem 4.3 (c) we have

$$\begin{aligned} \left\langle \min_{m \in S} (n+1)g(m) - 1 \right\rangle &\geq \left\langle \min_{m \in S} n \, f(m) \right\rangle \\ &= \left\langle n \, f(n) \right\rangle = \left\langle A(n) + B(n) \right\rangle \\ &= \left\langle (n+1)g(n) - 1 \right\rangle \\ &\geq \left\langle \min_{m \in S} (n+1)g(n) - 1 \right\rangle . \end{aligned}$$

Consequently, $\left\langle \min_{m \in S} (n+1)g(m) - 1 \right\rangle = \left\langle \min_{m \in S} n \, f(m) \right\rangle$ and part (c) is proved.

Theorem 4.6. Parts (a) and (b) of Theorem 4.5 are not sufficient conditions for equality to hold in parts (a) and (d) respectively of Theorem 4.3 nor is part (c) of Theorem 4.5 a necessary condition for equality to hold in part (c) of Theorem 4.3.

Proof: We will give examples which will establish Theorem 4.6. In Example 3 with $n = 12$ we have

$$\min_{m \in S} f(m) = f(7) = \frac{4}{7},$$

and

$$\min_{m \in S^*} f(m) = f(12) = \frac{9}{12} = \frac{3}{4}.$$

Thus, $\left\langle \min_{m \in S} f(m) \right\rangle = \left\langle \min_{m \in S^*} f(m) \right\rangle = 1$ and the equality of part (a) of Theorem 4.5 holds. However,

$$\min_{m \in S} n\, f(m) = 12 f(7) = \frac{48}{7},$$

and

$$\min_{m \in S^*} n\, f(m) = 12 f(12) = 9.$$

Then $\left\langle \min_{m \in S} n\, f(m) \right\rangle = 7 \neq \left\langle \min_{m \in S^*} n\, f(m) \right\rangle = 9$. Thus the equality of part (a) of Theorem 4.5 is not a sufficient condition for equality to hold in part (a) of Theorem 4.3.

Similarly, in Example 3 with $n = 12$ we have,

$$\min_{m \in S} g(m) = g(7) = \frac{5}{8},$$

and

$$\min_{m \in S^*} g(m) = g(12) = \frac{10}{13}.$$

Thus, $\left\langle \min_{m \in S} g(m) \right\rangle = \left\langle \min_{m \in S^*} g(m) \right\rangle = 1$ and the equality of part (b) of Theorem 4.5 holds. However,

$$\min_{m \in S} (n+1)g(m) - 1 = 13g(7) - 1 = \frac{57}{8},$$

and

$$\min_{m \in S^*} (n+1)g(m) - 1 = 13g(12) - 1 = 9.$$

Then, $\left\langle \min_{m \in S} (n+1)g(m) - 1 \right\rangle = 8 \neq \left\langle \min_{m \in S^*} (n+1)g(m) - 1 \right\rangle = 9$ so that the equality of part (b) of Theorem 4.5 is not a sufficient condition for equality to hold in part (d) of Theorem 4.3.

Finally, in Example 4 with $n = 4$ we have

$$\min_{m \in S} n\, f(m) = 4f(3) = 4\left(\frac{2}{3}\right) = \frac{8}{3},$$

and

$$\min_{m \in S} (n+1)g(m) - 1 = 5g(3) - 1 = 5\left(\frac{3}{4}\right) - 1 = \frac{11}{4}$$

so that $\left\langle \min_{m \in S} n\, f(m) \right\rangle = \left\langle \min_{m \in S} (n+1)g(m) - 1 \right\rangle = 3$ and equality holds in part (c) of Theorem 4.3. However, $\min_{m \in S} n\, f(m) = \frac{8}{3}$ while $n\, f(n) = 4f(4) = 4$ so that $\left\langle \min_{m \in S} n\, f(m) \right\rangle \neq \left\langle n\, f(n) \right\rangle$. Hence, the equality of part (c) of Theorem 4.5 is not a necessary condition for equality to hold in part (c) of Theorem 4.3. This completes the proof of Theorem 4.6.

Theorem 4.7. There exist examples where each of the eight cases of equality and strict inequality hold in the following inequalities:

(a) $\left\langle \min_{m \in S} n\, f(m) \right\rangle \le \left\langle \min_{m \in S^*} n\, f(m) \right\rangle$;

(b) $\left\langle \min_{m \in S^*} n\, f(m) \right\rangle \le \left\langle \min_{m \in S^*} (n+1)\, g(m) - 1 \right\rangle$;

(c) $\left\langle \min_{m \in S} n\, f(m) \right\rangle \le \left\langle \min_{m \in S} (n+1)\, g(m) - 1 \right\rangle$;

(d) $\left\langle \min_{m \in S} (n+1)\, g(m) - 1 \right\rangle \le \left\langle \min_{m \in S^*} (n+1)\, g(m) - 1 \right\rangle$.

Proof: The examples used in the proof of Theorem 3.4 will also suffice for the proofs of the corresponding parts of Theorem 4.7.

CHAPTER V

Proof of Mann's Lemma by Mann's Method

In this chapter we prove a lemma in preparation for the proof of Mann's Theorem (Theorem 2.3). We introduce the following additional notation. For any set let $C(-1) = -1$ and for $n \geq m$ let $C(m, n) = C(n) - C(m)$. A gap of C is a positive integer not in C.

Mann's Lemma. Let $C = A + B$, $0 \in A$, $0 \in B$. For $n > 0$, $n \notin C$ there exists a number $m \notin C$ such that $m = n$ or $0 < m < \frac{n}{2}$ for which

$$(5.1)\quad \frac{C(n)+1}{n+1} \geq \frac{A(m)+B(m)+1}{m+1} + [C(n-m-1)+1 - \frac{C(n)+1}{n+1}(n-m)] \frac{1}{m+1} .$$

Proof: For any $n > 0$, $n \notin C$, let $n_1 < n_2 < \cdots < n_r = n$ be the gaps of C less than or equal to n and form the differences $d_i = n_r - n_i$, $1 \leq i \leq r$. Define a sequence of numbers $e_1, e_2, \ldots, e_J$ and construct the sequences of sets $B = B_0, B_1, \ldots, B_J$ and $C = C_0, C_1, \ldots, C_J$ according to the following rules.

Rule 5.1. For $1 \leq j \leq J$, e_j is the least number in B_{j-1} for which there are integers $a \in A$ and $n_s, n_t \notin C_{j-1}$ such that

$$(5.2)\quad a + e_j + d_s = n_t .$$

(If there exists no $e_1 \in B_0$ let $J = 0$. Then the following lemmas leading to the proof of Mann's Lemma are still true.)

Rule 5.2. The number $e_j + d_s \in B_j^*$ if and only if equation (5.2) holds.

Rule 5.3. For $1 \leq j \leq J$ define $B_j = B_{j-1} \cup B_j^*$ and $C_j = A + B_j$.

Rule 5.4. For no numbers $n_s, n_t \notin C_J$ does $a + e_{J+1} + d_s = n_t$ for any $a \in A$, $e_{J+1} \in B_J$.

The construction places the numbers $n_s, n_t \notin C_{j-1}$ into the set C_j if equation (5.2) holds. Thus n_s and n_t are not available choices of gaps of C_j which will satisfy equation (5.2) in the construction of C_{j+1}. Therefore the construction continues until we either place every n_i, $1 \leq i \leq r$, into some set C_j, $1 \leq j \leq J$, or else for any n_i which are left it is not possible to find an a and an e satisfying equation (5.2).

We now prove several lemmas concerning the construction which will enable us to complete the proof of Mann's Lemma.

Lemma 5.1. The integer n_r is not in C_j, $0 \leq j \leq J$.

Proof. The integer n_r is not in C_0 by hypothesis. Assume $n_r \in C_j$, $n_r \notin C_{j-1}$, $1 \leq j \leq J$. Equation (5.2) holds if and only if $a + e_j + d_t = n_s$. Hence equation (5.2) implies here that $a + e_j + d_r = n_u$ where $n_u \notin C_{j-1}$. Since $d_r = 0$ and $e_j \in B_{j-1}$, then $n_u = a + e_j \in A + B_{j-1} = C_{j-1}$ which is a contradiction.

Lemma 5.2. The sets B_{j-1} and B_j^* are disjoint, $1 \leq j \leq J$.

Proof. Since B_j^* exists then equation (5.2) holds. Assume B_{j-1} and B_j^* have an element in common. Then for some $b^* \epsilon B_{j-1}$ and some $e_j + d_s \epsilon B_j^*$ we have $b^* = e_j + d_s$. Hence $n_t = a + e_j + d_s = a + b^*$. Since $b^* \epsilon B_{j-1}$ we have $n_t \epsilon C_{j-1}$. On the other hand $n_t \not\epsilon C_{j-1}$ since equation (5.2) holds. This is a contradiction.

Lemma 5.3. The numbers e_j form a non-decreasing sequence. That is, $e_1 \leq e_2 \leq \cdots \leq e_J$.

Proof. If $1 \leq j \leq J - 1$ we have $e_{j+1} \epsilon B_j$. If $e_{j+1} \epsilon B_{j-1}$ then $e_j \leq e_{j+1}$ by the minimal property of e_j stated in Rule 5.1. If $e_{j+1} \epsilon B_j^*$ then $e_{j+1} = e_j + d_i$ so that $e_j \leq e_{j+1}$.

For the remainder of the proof of Mann's Lemma denote the least gap of C_J by n_s. Since $n_r \not\epsilon C_J$ we have $1 \leq n_s \leq n_r$.

Lemma 5.4. $C_J(n_r - n_s - 1, n_r) = n_s$.

Proof. Let n_i be any gap of C where $n_r - n_s - 1 < n_i < n_r$. Then $0 \leq n_i + n_s - n_r < n_s$ and by our choice of n_s we have $n_i + n_s - n_r \epsilon C_J$. Hence $n_i + n_s - n_r = a + b'$, $b' \epsilon B_J$, that is $n_s - a = b' + d_i$. Since $n_s \not\epsilon C_J$, by Rule 5.4 we have $n_i \epsilon C_J$. Hence every integer in the interval $n_r - n_s - 1 < x < n_r$ is in C_J and since $n_r \not\epsilon C_J$ we have

$$C_J(n_r - n_s - 1, n_r) = n_r - (n_r - n_s - 1) - 1 = n_s.$$

Lemma 5.5. $C_J(n_r - n_s - 1, n_r) - C(n_r - n_s - 1, n_r) = B_J(n_s) - B(n_s)$.

Proof. For $J = 0$ the lemma is trivially true. Hence suppose $J \geq 1$. For each $j (1 \leq j \leq J)$ we have that $B_j = B_{j-1} \cup B^*_j$ and that $B_{j-1} \cap B^*_j$ is empty. Hence $B_j(n_s) = B_{j-1}(n_s) + B^*_j(n_s)$, and so

$$B_J(n_s) - B(n_s) = \sum_{j=1}^{J} B^*_j(n_s) .$$

We prove the lemma by establishing a biunique correspondence between gap n_t of C, where $n_t \in C_J$ and $n_r - n_s - 1 < n_t \leq n_r$, and integers $e_m + d_t$, where $e_m + d_t \in B^*_m$ and $0 < e_m + d_t \leq n_s$.

If $n_t \in C_J$, then for some $u, m (1 \leq u \leq r, 1 \leq m \leq J)$ we have $a + e_m + d_u = n_t$ where $n_t, n_u \notin C_{m-1}$ and $n_t, n_u \in C_m$. Then $a + e_m + d_t = n_u$ and so $e_m + d_t \in B^*_m$. Conversely if $e_m + d_t \in B^*_m$ $(1 \leq m \leq J)$, then for some $u (1 \leq u \leq r)$ we have $a + e_m + d_t = n_u$ where $n_u, n_t \notin C_{m-1}$ and $n_u, n_t \in C_m$. Since $C_m \subseteq C_J$ then $n_t \in C_J$.

Now assume that $0 < e_j + d_t \leq n_s$. Then $e_j + n_r - n_t \leq n_s$ and so $n_t \geq n_r - n_s + e_j \geq n_r - n_s > n_r - n_s - 1$. By definition $n_t \leq n_r$.

Finally assume $n_r - n_s - 1 < n_t \leq n_r$. Then corresponding to n_t we have $e_m + d_t \in B^*_j$ for some $m (1 \leq m \leq J)$. Since $n_r \notin C_J$, then $n_t < n_r$ and so $d_t > 0$. Hence $e_m + d_t > 0$. Suppose $e_m + d_t > n_s$. Then $e_m > n_s - d_t = a + b^*$, $b^* \in B_J$. If $m \leq j \leq J$, then $b^* \notin B^*_j$ for if $b^* \in B^*_j$, then by Lemma 5.3 we have $e_j \geq e_m > a + b^* = a + e_j + d_i \geq e_j$, which is a contradiction. Hence since $b^* \in B_J$

we must have $b^* \epsilon B_{m-1}$. From the manner in which $e_m + d_t$ is constructed from n_t we have $n_t \notin C_{m-1}$. Also $n_s \notin C_{m-1}$ since $n_s \notin C_J$. Finally since $a + b^* + d_t = n_s$ we have from Rule 5.1 that $e_m \leq b^*$ which contradicts $e_m > a + b^* \geq b^*$. Hence $e_m + d_t \leq n_s$. This completes the proof.

We are now prepared for the proof of Mann's Lemma.

By Lemma 2.1 we have $n_s \geq A(n_s) + B_J(n_s) + 1$. By Lemma 5.4 this becomes $C_J(n_r - n_s - 1, n_r) \geq A(n_s) + B_J(n_s) + 1$. Using Lemma 5.5 we obtain

$$C(n_r - n_s - 1, n_r) \geq A(n_s) + B(n_s) + 1,$$

or finally

$$C(n_r) \geq A(n_s) + B(n_s) + 1 + C(n_r - n_s - 1) \tag{5.3}$$

By adding 1 to both members of inequality (5.3) and dividing by $n_s + 1$ we obtain

$$\frac{C(n) + 1}{n_s + 1} \geq \frac{A(n_s) + B(n_s) + 1}{n_s + 1} + \frac{C(n_r - n_s - 1) + 1}{n_s + 1}.$$

Upon multiplying the left member by unity in the form of

$$1 = \frac{n_s + 1 + n_r - n_s}{n_r + 1}$$

we have

$$[C(n_r)+1]\left[\frac{n_s+1+n_r-n_s}{(n_s+1)(n_r+1)}\right] \geq \frac{A(n_s) + B(n_s) + 1}{n_s + 1} + \frac{C(n_r-n_s-1) + 1}{n_s + 1},$$

or

$$[C(n_r)+1]\left[\frac{1}{n_r+1}+\frac{n_r-n_s}{(n_s+1)(n_r+1)}\right]\geq\frac{A(n_s)+B(n_s)+1}{n_s+1}+\frac{C(n_r-n_s-1)+1}{n_s+1}.$$

By distributing $[C(n_r)+1]$ in the left hand member and transposing we obtain

$$\frac{C(n_r)+1}{n_r+1}\geq\frac{A(n_s)+B(n_s)+1}{n_s+1}+\frac{C(n_r-n_s-1)+1}{n_s+1}-\frac{[C(n_r)+1](n_r-n_s)}{(n_s+1)(n_r+1)}.$$

By factoring $\frac{1}{n_s+1}$ from the last two terms of the right hand member we obtain

$$\frac{C(n_r)+1}{n_r+1}\geq\frac{A(n_s)+B(n_s)+1}{n_s+1}+\left[C(n_r-n_s-1)+1-\frac{C(n_r)+1}{n_r+1}(n_r-n_s)\right]\frac{1}{n_s+1},$$

which is inequality (5. 1) of Mann's Lemma with $n_r = n$ and $n_s = m$.

If $n_s < n_r$, then by Lemma 5. 4 there are no gaps of C_J in the interval $n_r - n_s - 1 < x < n_r$. Hence $n_s < n_r - n_s$, or $n_s < \frac{n_r}{2}$, which completes the proof of Mann's Lemma.

CHAPTER VI

Proof of Mann's Lemma by Artin and Scherk's Method

In 1943 Emil Artin and Peter Scherk (1) published a paper in which they establish the inequality

$$(6.1) \qquad C(n) \geq A(m-1) + B(m-1) + C(n-m)$$

for $n \geq m$; $n, m \notin C$ without the assumption $0 \in A$, $0 \in B$.

By a similar construction to that of Chapter V they obtained a number of lemmas preliminary to inequality (6.1) which are quite similar to the lemmas of Chapter V. This similarity in the lemmas plus the similarity of inequality (6.1) to inequality (5.3) of Chapter V prompted us to attempt a proof of Mann's Lemma by using Artin and Scherk's method but with the additional assumption of $0 \in A$, $0 \in B$.

The following proof of Mann's Lemma will be found to be quite similar to the proof of Mann's Lemma given in Chapter V. For convenience we restate Mann's Lemma.

Mann's Lemma. Let $C = A + B$, $0 \in A$, $0 \in B$. For $n > 0$, $n \notin C$, there exists a number $m \notin C$ such that $m = n$ or $0 < m < \frac{n}{2}$, for which

$$(6.2) \qquad \frac{C(n)+1}{n+1} \geq \frac{A(m)+B(m)+1}{m+1} + \left[C(n-m-1)+1 - \frac{C(n)+1}{n+1}(n-m)\right] \frac{1}{m+1} .$$

Proof: For any $n > 0$, $n \notin C$ let $n_1 < n_2 < \cdots < n_r = n$ be the gaps of C less than or equal to n_r and form the differences $d_i = n_r - n_i$, $1 \leq i \leq r$. Define a sequence of numbers $e_1, e_2, \ldots, e_J$ and construct the sequences of sets $B = B_0, B_1, \ldots, B_J$ and $C = C_0, C_1, \ldots, C_J$ according to the following rules.

Rule 6.1. For $1 \leq j \leq J$, e_j is the least number in B_{j-1} for which there are integers $a \in A$ and $n_s, n_t \notin C_{j-1}$ such that

(6.3) $$a + e_j + d_s = n_t$$

(If there exists no $e_1 \in B_0$ let $J = 0$. Then the following lemmas leading to the proof of Mann's Lemma are still true.)

Rule 6.2. For $1 \leq j \leq J$ let

$$C_j^* = \{n_s : n_s \text{ satisfies equation (6.3)}\}$$
$$B_j^* = \{e_j + d_s : n_s \in C_j^*\} .$$

Rule 6.3. For $1 \leq j \leq J$ define B_j and C_j by

$$B_j = B_{j-1} \cup B_j^* \text{ and } C_j = C_{j-1} \cup C_j^*.$$

Rule 6.4. For no numbers $n_s, n_t \notin C_J$ does $a + e_{J+1} + d_s = n_t$ for any $a \in A$, $e_{J+1} \in B_J$.

We now state and prove several lemmas concerning the construction which will enable us to complete the proof of Mann's Lemma.

Lemma 6.1. For $0 \leq j \leq J$ the set C_j is contained in $A + B_j$.

Proof: For $j = 0$, $C_0 = A + B_0$, and the lemma holds trivially. Assume the lemma is true for $j = k$ $(0 \leq k < J)$. Let c be any element of C_{k+1}. If $c \in C_k$, then $c \in A + B_k \subset A + B_{k+1}$. If $c \in C^*_{k+1}$, then by Rule 6.1 and 6.2 there exists an $a \in A$ and an $n_i \notin C_k$ such that $a + e_{k+1} + d_i = c$ where $e_{k+1} + d_i \in B^*_{k+1}$. Thus $c \in A + B^*_{k+1} \subset A + B_{k+1}$ which completes the proof of the lemma.

Lemma 6.2. If $a + b \in A + B_j$ and $0 \leq a + b \leq n_r$, then $a + b \in C_j$ $(0 \leq j \leq J)$.

Proof. For $j = 0$ the lemma is trivial. Assume the lemma is true for $j = k$ $(0 \leq k < J)$. Let $a + b \in A + B_{k+1}$, where $a \in A$ and $b \in B_{k+1}$, and $0 \leq a + b \leq n_r$. If $b \in B_k$ then $a + b \in C_k \subset C_{k+1}$. If $b \in B^*_{k+1}$ then $a + b = a + e_{j+1} + d_i$ where $n_i \notin C_k$. If $a + b \notin C_k$ then by Rule 6.2 we have $a + b \in C^*_{k+1} \subset C_{k+1}$, and the proof of the lemma is complete.

Lemma 6.3. The integer n_r is not in C_j, $0 \leq j \leq J$.

Proof. The number n_r is not in C_0 by hypothesis. Assume $n_r \in C_j$, $n_r \notin C_{j-1}$, $1 \leq j \leq J$. Then equation (6.3) holds so that $a + e_j + d_i = n_r$ for $n_r, n_i \notin C_{j-1}$. Equivalently $a + e_j = n_i$. Also $a + e_j \leq n_r$ since $d_i \geq 0$. Since $a + e_j \in A + B_{j-1}$ we have by Lemma 6.2 that $n_i = a + e_j \in C_{j-1}$, which is a contradiction. Hence $n_r \notin C_j$.

Lemma 6.4. The sets B^*_j and B^*_{j-1} are disjoint $(1 \leq j \leq J)$.

Proof. Let $e_j + d_i$ be any element of B^*_j. Then for some $a \epsilon A$ and $n_k \notin C_{j-1}$ we have $e_j + d_i = n_k - a$. If $n_k - a \epsilon B^*_{j-1}$ then $a + (n_k - a) = n_k \epsilon A + B_{j-1}$. Since $1 \leq n_k \leq n_r$ we have $n_k \epsilon C_{j-1}$ by Lemma 6.2, contrary to $n_k \notin C_{j-1}$. Thus $e_j + d_i \notin B^*_{j-1}$ and B^*_j and B^*_{j-1} are disjoint.

Lemma 6.5. The integers e_j form a non-decreasing sequence. That is, $e_1 \leq e_2 \leq \cdots \leq e_J$.

Proof. If $1 \leq j \leq J - 1$ we have $e_{j+1} \epsilon B_j$. If $e_{j+1} \epsilon B_{j-1}$ then $e_j \leq e_{j+1}$ by the minimal property of e_j stated in Rule 6.1. If $e_{j+1} \epsilon B^*_j$ then $e_{j+1} = e_j = d_i$ so that $e_j \leq e_{j+1}$.

For the remainder of the proof of Mann's Lemma denote the least gap of C_J by n_s. Since $n_r \notin C_J$ we have $1 \leq n_s \leq n_r$.

Lemma 6.6. For $1 \leq j \leq J$ we have $C^*_j(n_r - n_s - 1, n_r) = B^*_j(n_s)$.

Proof. Since $n_r \notin C^*_j$, then $C^*_j(n_r - n_s - 1, n_r)$ represents the number of integers $n_i \epsilon C^*_j$ such that $n_r - n_s - 1 < n_i < n_r$.

For each $n_i \epsilon C^*_j$ we let correspond $e_j + d_i \epsilon B^*_j$. This is a biunique correspondence between C^*_j and B^*_j by Rules 6.1 and 6.2. We have $0 < e_j + d_i \leq n_s$, for suppose $e_j + d_i > n_s$. Then $e_j > n_s - d_i$. Now $d_i > 0$ for if $d_i = 0$ we have $n_r - n_i = 0$ or $n_r = n_i$. But $n_r \notin C^*_j$ which contradicts $n_i \epsilon C^*_j$. Hence $n_s - d_i < n_s$ so that $n_s - d_i \epsilon C_J$. Thus $e_j > n_s - d_i = a + b^*$, $b^* \epsilon B_J$. Since $e_j > a + b^*$ we must have $b^* \notin B^*_j$ for if $b^* \epsilon B^*_j$ we have $e_j > a + e_j + d_t$ which is not possible. Lemma 6.5 extends the exclusion of

b^* to all sets $B^*_m, j+1 \leq m \leq J$. But since $b^* \in B_J$ we must have $b^* \in B_{j-1}$. By Rule 6.1 we have $e_j \leq b^*$ which contradicts $e_j > a + b^*$.

Also since $0 < e_j + d_i \leq n_s$, then $e_j + n_r - n_i \leq n_s$ and so $n_i \geq n_r - n_s + e_j \geq n_r - n_s > n_r - n_s - 1$. Furthermore $n_i < n_r$. Thus

$$C^*_j(n_r - n_s - 1, n_r) = B^*_j(n_s).$$

Lemma 6.7. $C_J(n_r - n_s - 1, n_r) = n_s$.

Proof. Let n_i be any integer such that $n_r - n_s - 1 < n_i < n_r$. Then $0 \leq n_i + n_s - n_r < n_s$. By our choice of n_s we have $n_i + n_s - n_r \in C_J$. By Lemma 6.1 we have $n_i + n_s - n_r = a + b'$, $b' \in B_J$, that is, $n_s - a = b' + d_i$. Since $n_s \notin C_J$ then Rule 6.4 tells us $n_i \in C_J$. Hence each n_i such that $n_r - n_s - 1 < n_i < n_r$ is in C_J and $n_r \notin C_J$ so that

$$\begin{aligned} C_J(n_r - n_s - 1, n_r) &= n_r - (n_r - n_s - 1) - 1 \\ &= n_s. \end{aligned}$$

Lemma 6.8. $B_J(n_s) - B(n_s) = C_J(n_r - n_s - 1, n_r) - C(n_r - n_s - 1, n_r)$.

Proof. By construction $B_J = B \cup B^*_1 \cup \ldots \cup B^*_J$ and since B^*_j and B^*_{j-1} are disjoint we have

$$B_J(n_s) = B(n_s) + B^*_1(n_s) + \cdots + B^*_J(n_s)$$

or

$$B_J(n_s) - B(n_s) = B^*_1(n_s) + \cdots + B^*_J(n_s).$$

Also by construction $C_J = C \cup C_1^* \cup \ldots \cup C_J^*$ where C_j^* and C_{j-1}^* are disjoint. Thus

$$C_J(n_r-n_s-1, n_r) - C(n_r-n_s-1, n_r) = C_1^*(n_r-n_s-1, n_r) + \cdots + C_J^*(n_r-n_s-1, n_r).$$

By Lemma 6.6 we obtain

$$C_J(n_r-n_s-1, n_r) - C(n_r-n_s-1, n_r) = B_J(n_s) - B(n_s),$$

which completes the proof of the lemma.

We are now prepared for the completion of the proof of Mann's Lemma.

By Lemma 2.1 we have $n_s \geq A(n_s) + B_J(n_s) + 1$.
By Lemma 6.7 this becomes

$$C_J(n_r-n_s-1,\ n_r) \geq A(n_s) + B_J(n_s) + 1.$$

Using Lemma 6.8 we obtain

$$C(n_r-n_s-1,\ n_r) \geq A(n_s) + B(n_s) + 1,$$

or finally,

$$C(n_r) \geq A(n_s) + B(n_s) + 1 + C(n_r-n_s-1).$$

But this is just inequality 5.3 which was seen in Chapter V to imply

$$\frac{C(n_r)+1}{n_r+1} \geq \frac{A(n_s)+B(n_s)+1}{n_s+1} + [C(n_r - n_s - 1) + 1 - \frac{C(n_r)+1}{n_r+1}(n_r-n_s)]\ \frac{1}{n_s+1},$$

which is inequality (6. 2) of Mann's Lemma with $n_r = n$ and $n_s = m$.

If $n_s < n_r$ then by Lemma 6. 7 there are no gaps of C_J in the interval $n_r - n_s - 1 < x < n_r$. Hence $n_s < n_r - n_s$, or $n_s < \frac{n_r}{2}$, which completes the proof of Mann's Lemma.

CHAPTER VII

Proof of Mann's Theorem

A consequence of Mann's Lemma is the following possibly more interesting result.

Mann's Theorem (Theorem 2.3). Let $C = A + B$, $0 \in A$, $0 \in B$. For $n > 0$ either $C(n) = n$, or $C(n) < n$ and there exist numbers m, m_1 such that $m \notin C$, $0 < m \leq n$, $m_1 \notin C$, $0 < m_1 \leq \max(m, n - m - 1)$, for which

$$(7.1) \quad \frac{C(n)+1}{n+1} \geq \frac{A(m)+B(m)+1}{m+1} + \left| \frac{C(n)+1}{n+1} - \frac{C(m_1)+1}{m_1+1} \right| .$$

Proof: The proof is by induction. Let $n = 1$. If $1 \in C$ we have $C(1) = 1$. If $1 \notin C$ then (7.1) holds for $m_1 = m = 1$ since $C(1) = A(1) = B(1) = 0$. Let k be an integer greater than 1. We assume that Mann's Theorem is valid if $1 \leq n < k$ and show that it is valid for $n = k$.

If $C(k) = k$ the theorem holds trivially, so we assume $C(k) < k$. Let k^* be the least gap of C. Then $k^* \leq k$. If $k^* = k$ then in Mann's Lemma we have $m = k$ and so $\frac{C(k)+1}{k+1} \geq \frac{A(k)+B(k)+1}{k+1}$. Thus Mann's Theorem holds by choosing $m = m_1 = k$. Hence we assume $k^* < k$.

Our hypotheses, including the inductive hypothesis, now include $C(k) < k$; $1 \leq k^* < k$ and

$$(7.2)\qquad \frac{C(k')+1}{k'+1} \geq \frac{A(m') + B(m')+1}{m'+1} + \left| \frac{C(k')+1}{k'+1} - \frac{C(m_1')+1}{m_1'+1} \right| ,$$

where $m' \notin C$, $0 < m' \leq k'$; $m_1' \notin C$, $0 < m_1' \leq \max(m', k' - m' - 1)$, for all k' such that $k^* \leq k' < k$. We now consider two cases.

Case 1. For some k' we have $\dfrac{C(k)+1}{k+1} \geq \dfrac{C(k')+1}{k'+1}$.

Then

$$\frac{C(k)+1}{k+1} = \left| \frac{C(k)+1}{k+1} - \frac{C(k')+1}{k'+1} \right| + \frac{C(k')+1}{k'+1} .$$

By inequality (7. 2) we obtain

$$\frac{C(k)+1}{k+1} \geq \left|\frac{C(k)+1}{k+1} - \frac{C(k')+1}{k'+1}\right| + \frac{A(m')+B(m')+1}{m'+1} + \left| \frac{C(k')+1}{k'+1} - \frac{C(m_1')+1}{m_1'+1} \right|$$

$$\geq \frac{A(m')+B(m')+1}{m'+1} + \left|\frac{C(k)+1}{k+1} - \frac{C(m_1')+1}{m_1'+1}\right| ,$$

where $m' \notin C$, $0 < m' \leq k' < k$, $m_1' \notin C$, and $0 < m_1' \leq \max(m', k' - m' - 1) \leq \max(m', k - m' - 1)$. Thus the theorem is true for $n = k$.

Case 2. For all k' we have $\dfrac{C(k)+1}{k+1} < \dfrac{C(k')+1}{k'+1}$. If we assume $k \in C$, then $C(k) = C(k-1) + 1$. Hence since $C(k) < k$ we have

$$\begin{aligned}(k+1)[C(k-1) + 1] &= (k+1)C(k) \\ &= k\,C(k) + C(k) \\ &< k\,C(k) + k = k[C(k) + 1],\end{aligned}$$

or

$$\frac{C(k-1)+1}{(k-1)+1} < \frac{C(k)+1}{k+1} .$$

But since $k-1$ is a value of k' this contradicts the assumption of this Case 2 so that $k \notin C$.

Since $k \notin C$, by Mann's Lemma there exists an $m^* \notin C$, $m^* = k$ or $0 < m < \frac{k}{2}$, such that

$$(7.3) \quad \frac{C(k)+1}{k+1} \geq \frac{A(m^*)+B(m^*)+1}{m^*+1} + \left[C(k-m^*-1)+1-\frac{C(k)+1}{k+1}(k-m^*)\right] \frac{1}{m^*+1}.$$

If $m^* = k$ we have

$$\frac{C(k)+1}{k+1} \geq \frac{A(k)+B(k)+1}{k+1} ,$$

and Mann's Theorem holds with $m = m_1 = k$.

If $0 < m^* < \frac{k}{2}$, then $k - m^* - 1 \geq m^*$. Since m^* is an integer not in C there exists a largest integer not in C which does not exceed $k - m^* - 1$. Denote this number by m_1^* so that $m^* \leq m_1^* \leq k - m^* - 1$, $m_1^* \notin C$. Then by definition of m_1^* we have

$$(7.4) \qquad C(k-m^*-1) - C(m_1^*) = (k-m^*-1) - m_1^*.$$

Now $m_1^* + 1 \geq C(m_1^*) + 1$ so that

$$(k-m^*-1-m_1^*)(m_1^*+1) \geq (k-m^*-1-m_1^*)[C(m_1^*)+1]$$

or

$$(m_1^*+1)[C(m_1^*)+1] + (k-m^*-1-m_1^*)(m_1^*+1) \geq (k-m^*)[C(m_1^*)+1].$$

Factoring $m_1^* + 1$ from the left hand member we obtain

$$(m_1^*+1)[C(m_1^*)+1 + (k-m^*-1-m_1^*)] \geq (k-m^*)[C(m_1^*)+1].$$

Thus by equation (7.4),

$$(m_1^*+1)[C(k-m^*-1)+1] \geq (k-m^*)[C(m_1^*)+1],$$

so that

$$\text{(7.5)} \qquad \frac{C(k-m^*-1)+1}{k-m^*} \geq \frac{C(m_1^*)+1}{m_1^*+1}.$$

Since $0 < m^* < \frac{k}{2}$, then

$$\text{(7.6)} \qquad \frac{k-m^*}{m^*+1} \geq 1.$$

Now from inequality (7.3) we have

$$\frac{C(k)+1}{k+1} \geq \frac{A(m^*) + B(m^*) + 1}{m^* + 1} + \left[\frac{C(k-m^*-1)+1}{k-m^*} - \frac{C(k)+1}{k+1}\right] \frac{k-m^*}{m^*+1}.$$

By inequalities (7.5) and (7.6) this becomes

$$\frac{C(k)+1}{k+1} \geq \frac{A(m^*) + B(m^*) + 1}{m^* + 1} + \left[\frac{C(m_1^*)+1}{m_1^*+1} - \frac{C(k)+1}{k+1}\right] \frac{k-m^*}{m^*+1}.$$

Since $k^* \leq m_1^* \leq k - m^* - 1 < k$, then

$$\frac{C(m_1^*)+1}{m_1^*+1} - \frac{C(k)+1}{k+1} > 0$$

by our assumption for this Case 2. Hence, because of inequality (7.6) we have

$$\frac{C(k)+1}{k+1} \qquad \frac{A(m^*)+B(m^*)+1}{m^*+1} + \left| \frac{C(k)+1}{k+1} - \frac{C(m_1^*)+1}{m_1^*+1} \right| .$$

Furthermore, $0 < m^* < \frac{k}{2} < k$, $m^* \notin C$, $m_1^* \notin C$, and $0 < m_1^* \leq \max(m^*, k - m^* - 1)$, so that Mann's Theorem is again true for $n = k$. This completes the proof by induction of Mann's Theorem. Note that the inductive hypothesis was not used in Case 2.

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