AN ABSTRACT OF THE THESIS OF

Daniel J. O'Regan for the degree of Doctor of Philosophy in Mathematics presented on April 25, 1985

Title: Initial And Boundary Value Problems Via Topological Methods

Redacted for Privacy

Abstract approved: [redacted]

John Walter Lee

In this thesis a relatively new topological technique, due to A. Granas, called Topological Transversality is used to obtain existence theorems for initial and boundary value problems in a variety of settings. This fixed point result is based on the notions of an essential map and on a priori bounds on solutions.

Initial And Boundary Value Problems Via Topological Methods

by

Daniel J. O'Regan

A THESIS
submitted to
Oregon State University

in partial fulfillment of
the requirements for the
degree of

Doctor of Philosophy

Completed April 25, 1985

Commencement June 1985

APPROVED:

Redacted for Privacy

Professor of Mathematics in charge of major

Redacted for Privacy

Chairman of Department of Mathematics

Redacted for Privacy

Dean of Graduate School

Date thesis is presented April 25, 1985

Typed by Naomi Normand for Daniel J. O'Regan

ACKNOWLEDGEMENTS

To begin with I would like to thank two people without whom my mathematical career would not have been possible. Firstly, many thanks to Mr. Liam Byrnes of Coláiste Chríost Rí, Cork, Ireland for taking the time and interest to help me in my early mathematical training. He was my first mathematics teacher and I have yet to meet anyone his equal in the teaching of mathematics. Secondly, and most importantly to my mother for believing in me and giving me the confidence to reach out and fulfill my dreams.

During my four year stay here at Oregon State University I've had much help from the faculty of the Mathematics Department. In particular, I would like to thank Professor's Ronald Guenther and Don Solmon for their kindness and invaluable assistance. Finally, and most importantly many many thanks to my major professor John Lee without whose support and encouragement this thesis would never have been completed. I greatly appreciate the time and effort he put in during the research and writing of the thesis. If I have become a better mathematician it is due totally to John Lee. He will always have a special place in my memory of Corvallis.

Many thanks to my friends in Ireland and Corvallis, who helped and encouraged me while I was working on this degree; in particular Brian Kelly, Scott Lewis, Naomi Normand, Henry O'Keeffe, Dale Rohm, Jimmy Ryan and Steve Smith. A special thank you to Naomi who did a wonderful job in typing the thesis. Above all these very little would have been accomplished without the love and support of my family in Ireland. Words cannot express my love and gratitude; however, I am very proud to dedicate this thesis to my father Cornelius, to my mother Eileen, to my sisters Mary Rose and Marguerita and to my brothers Francis and Gerard.

TABLE OF CONTENTS

LIST OF FIGURES

PREFACE

In his 1976 paper "Sur La méthode de continuité de Poincaré" A. Granas provided a new approach to the continuity method for establishing fixed points of certain mappings. This new theory, now called Topological Transversality, was utilized in the papers of Granas, Guenther and Lee [15], [16], [17] and [18] to study existence questions for nonlinear boundary value problems. In this thesis we extend further these ideas and show how powerful and natural Topological Transversality is for solving a variety of problems.

We begin in Chapter II by using Topological Transversality to examine the dependence of the interval of existence for an initial value problem upon its initial data and the nonlinearity in the differential equations. Stronger and more applicable existence theorems than the local existence theorems that are available in the literature are proven. These results enable us to read off directly from the differential equation an interval of existence of a solution and in many cases it will be maximal as our theory shows.

In Chapter III the nonlinear differential equation $y'' = f(x,y,y')$, $0 \leq x < \infty$ with appropriate boundary conditions is studied. Our treatment involves extending results of Granas, Guenther and Lee [15], [16] and [18] on finite interval boundary value problems with f satisfying Bernstein type growth conditions. We also examine an important application which occurs in the theory of semiconductor devices.

The initial suggestion for examining ordinary differential equations in the complex domain was Professor John Lee's and we pursue this in Chapter IV. Although initial value problems in the complex domain have been studied widely, very little is known on intervals of existence of a solution. The results of this chapter enable us to read off immediately from the differential equation an interval of existence of a solution, although this interval may not be maximal. Furthermore, we introduce boundary value problems in the complex domain and obtain existence theorems for such problems.

While working on this thesis I had many invaluable discussions with Professor Ronald Guenther and it was he who suggested looking for "weak solutions" via Topological Transversality to boundary value problems of the form $y'' = f(t,y,y')$, $t \in [0,1]$, with y satisfying suitable boundary conditions, where $f:[0,1] \times R^2 \to R$ is discontinuous. In Chapter V we examine such problems with f satisfying Caratheodory Conditions and obtain solutions to the above problem in Sobolev spaces. Our analysis is based on a priori bounds and known results on Nemysky operators and Sobolev spaces. Furthermore, we also obtain weak solutions to initial value problems.

In Chapter VI third order boundary value problems are studied. The chapter is divided into two parts, the first of which extends the Bernstein theory and results of Granas, Guenther and Lee [15], [16] on second order boundary value problems. Many of the results we obtain are rather specialized. In part two, by examining different types of monotonicity and growth conditions, we obtain existence theorems for a wide class of new problems.

Over the last ten years ordinary differential equations in abstract spaces have become very popular and in Chapter VII we suggest a new method, via Topological Transversality, for examining nonlinear differential equations in Hilbert spaces. Furthermore, we show how the analysis in this chapter can be used to obtain existence of solutions to certain integro-differential equations. The suggestion for examining such problems was again provided by Professor John Lee.

Initial And Boundary Value Problems Via Topological Methods

I. Preliminaries

1.1 Notation

In this chapter we formulate the basic ideas which will be used throughout this thesis. The following standard notation is used: $R = (-\infty,\infty)$ will denote the real line and R^n the n-dimensional Euclidean space. By $C^K = C^K(S)$, $K \geq 0$ an integer, and $C^o = C(S)$ we denote the functions which are K times continuously differentiable on S. If $x = x(t)$ is a real valued function defined on S, we set

$$|x|_o = \sup_{t \in S} |x(t)|$$

and for functions $x(t) \in C^K$, we set

$$|x|_K = \max\{|x|_o, |x'|_o, \ldots\ldots, |x^{(K)}|_o\}.$$

Finally let $C^K(S)$ denote the Banach Space of functions for which $|x|_K < \infty$. See [4], [30] for details.

Also if $u = (u_1,\ldots\ldots,u_n)$, $v = (v_1,\ldots\ldots,v_n)$ are vectors in R^n, we set

$$\langle u,v\rangle = \sum_{i=1}^{n} u_i\, v_i \quad \text{and} \quad |u| = (\langle u,u\rangle)^{1/2}.$$

Further, if $u:S \to R^n$, where $S \subset R$, is a vector valued function of t, we set

$$\|u\|_o = \max_j \sup_{t \in S} |u_j(t)|, \quad j = 1, \ldots\ldots, n$$

and if u is in C^K

$$\|u\|_K = \max_{\ell} \|u^{(\ell)}\|_o, \quad \ell = 0, 1, \ldots\ldots, K.$$

If B is a set of boundary or initial conditions then C^K denotes the subset of functions in C^K which satisfy these boundary or initial conditions.

1.2 Theorems from Ordinary Differential Equations

We begin by collecting together a few standard facts about ordinary differential equations which will be used in various parts of the thesis.

Suppose that $f:[0,1] \times R^n \to R$ is a continuous function. Consider the n^{th} order initial value problem

$$1.1) \begin{cases} y^{(n)}(t) = f(t, y(t), y'(t), \ldots\ldots, y^{(n-1)}(t)), t \in [0,1] \\ y(t_o) = y_o, y'(t_o) = y_o^1, \ldots\ldots, y^{(n-1)}(t_o) = y_o^{n-1}; t_o \in [0,1] \end{cases}$$

By a solution to (1.1) we mean a real valued function y which is n times continuously differentiable on $[0,1]$ which satisfies the differential equation and initial conditions.

<u>Theorem 1.1.</u> Let $a_o(t), \ldots\ldots, a_n(t) \neq 0$ be defined and continuous on $[0,1]$. Then the initial value problem

$$\begin{cases} Ly \equiv a_n(t)y^{(n)}(t)+a_{n-1}(t)y^{(n-1)}(t)+\ldots+a_o(t)y(t)=0 \\ y(t_o)=y_o,\ldots,y^{(n-1)}(t_o)=y_o^{n-1};t_o\in[0,1] \end{cases}$$

has a unique solution $y \in C^n[0,1]$.

As a consequence of Theorem 1.1 it is possible to construct n linearly independent solutions to $Ly = 0$ and thus we obtain the following result.

Theorem 1.2. Let $y_1(t),\ldots\ldots, y_n(t)$ be n linearly independent solutions to $Ly = 0$. Then every solution $y(t)$ of $Ly = 0$ is of the form

1.2) $$y(t) = C_1 \; y_1(t) + \ldots + C_n \; y_n(t)$$

for some choice of constants $C_1,\ldots, C_n$. For this reason we say that (1.2) is the general solution to $Ly = 0$.

1.3 Topological Transversality and A Priori Bounds

We begin with a description of the topological ideas and results which we shall use. For complete proofs and further details see Dugundji and Granas [11].

Let E be a normed linear space and $K \subset E$ a convex set. Also let $U \subset K$ be open in K and $\bar{U}$ and ∂U denote respectively the closure and boundary of U in K.

Definition 1.1 Suppose X is a metric space and $F:X \to K$ is a continuous map.

(i) F is compact if $F(X)$ is contained in a compact subset of K.

(ii) F is completely continuous if it maps bounded subsets in X into compact subsets of K.

We formulate the Schauder fixed point theorem without proof. See [11] for details.

<u>Theorem 1.3</u>. Suppose E is a normed linear space, $K \subset E$ a convex set and $F: K \to K$ a compact map. Then F has a fixed point in K i.e. there exists $x_o \in K$ such that $F(x_o) = x_o$.

Next we formulate a fixed point theory similar to the continuity method of Leray and Schauder which uses topological degree. Instead we use the notion of an essential map.

<u>Definition 1.2</u> Let K be a convex subset of a normed linear space E and $U \subset K$ be open in K.

(i) A compact map $F: \bar{U} \to K$ is called <u>admissible</u> if it is fixed point free on ∂U. The set of all such maps will be denoted by $K_{\partial U}(\bar{U}, K)$.

(ii) A map $F \in K_{\partial U}(\bar{U}, K)$ is <u>essential</u> if every compact map which agrees with F on ∂U has a fixed point in U. Otherwise F is <u>inessential</u>.

(iii) A homotopy $\{H_t : X \to K\}$, $0 \leqslant t \leqslant 1$, is said to be compact provided the map $H: X \times [0,1] \to K$ given by $H(x,t) = H_t(x)$ for $(x,t) \in X \times [0,1]$ is compact.

(iv) Two maps $F, G \in K_{\partial U}(\bar{U}, K)$ are <u>homotopic</u> if there is a compact homotopy $H_t : \bar{U} \to K$ for which $F = H_o$, $G = H_1$ and H_t is admissible for each t in $[0,1]$.

<u>Remark</u>. Every essential map has a fixed point in U.

The following theorem yields some simple examples of essential maps.

<u>Theorem</u> <u>1.4.</u> Let $p \in U$ and $F \in K_{\partial U}(\bar{U},K)$ be the constant map $F(x) = p$ for $x \in \bar{U}$. Then F is essential.

<u>Proof</u> Let $G:\bar{U} \to K$ be a compact map with $G = F$ on ∂U. Define

$$H(x) = \begin{cases} G(x), & x \in \bar{U} \\ p, & x \in K \setminus \bar{U}. \end{cases}$$

Now $H:K \to K$ is compact and so by Theorem 1.3 there exists $x_o \in K$ such that $H(x_o) = x_o$. However, by definition of H and the fact that $F = G$ on ∂U implies $x_o \in U$ and $x_o = H(x_o) = G(x_o)$. Thus G has a fixed point and F is essential.

Theorem 1.4 is a key element in the implementation of the Topological Transversality Theorem in obtaining existence theorems for initial and boundary value problems. We now state the Topological Transversality Theorem due to A. Granas without proof. Details of the proof can be obtained in Granas [14], Dugundji and Granas [11] and Granas, Guenther and Lee [17].

<u>Theorem</u> <u>1.5</u>. Suppose

(a) F and G are compact maps on $\bar{U} \to K$.
(b) G is essential in $K_{\partial U}(\bar{U},K)$.
(c) $H(x,t)$, $0 \leqslant t \leqslant 1$, is a compact homotopy joining F and G i.e. $H(x,0) = G(x)$, $H(x,1) = F(x)$.
(d) $H(x,t)$ is for each t, $0 \leqslant t \leqslant 1$, fixed point free on ∂U.

Then there exists at least one fixed point $u \in U$

such that $u = F(u)$.

Remark. Suppose we now let $K = E$. To apply Theorem 1.5 we let U be some ball centered at the origin of radius say R. Let $u_o \in U$ be a fixed point of G. In order to guarantee the existence of a fixed point $u_1 \in \bar{U}$ of F we need to know that all fixed points u_t of $u = H(u,t)$ lie in U i.e. we need an A Priori Bound $M < R$ such that

$$\|u_t\| \leq M \quad \text{for} \quad 0 \leq t \leq 1,$$

where $\|\cdot\|$ is the norm in E. Therefore the role of the a priori bound is to guarantee that ∂U is fixed point free for the operators $H(\cdot,t)$ and these bounds must be independent of t.

We will also use frequently the following theorems which we state without proof. See Rudin [29] and Royden [28] for details.

Theorem 1.6. (Bounded Inverse Theorem)

Suppose X and Y are Banach Spaces and L is a bounded linear transformation from X onto Y which is also one to one. Then L^{-1} is a bounded linear transformation of Y onto X.

Theorem 1.7. (Arzela Ascoli Theorem)

Let F be an equicontinuous family of real or complex valued functions on a seperable space X. Then each sequence $\langle f_n \rangle$ in F which is bounded at each point (of a dense subset) has a subsequence $\langle f_{n_K} \rangle$ which converges pointwise to a continuous function, the converence being uniform on each compact subset of X.

From Theorem 1.7 one can deduce:

Corollary 1.8. Let S be a compact metric space and M a subset of $C^K(S)$, $K \geq 0$ an integer. Then M is relatively compact in $C^K(S)$ (i.e. $\bar{M}$ is a compact subset of $C^K(S)$) if and only if M is bounded and equicontinuous.

1.4 Existence Theorems

In this section we indicate how the topological results of section 1.3 can be used to prove existence theorems for nonlinear boundary value problems. Similar results hold for nonlinear initial value problems.

Let B denote a set of n linear, homogeneous boundary conditions

$$V_i(y) = \sum_{j=0}^{n-1} [a_{ij} y^{(j)}(0) + b_{ij} y^{(j)}(1)] = 0, i = 1, 2, \ldots, n$$

and $(Ly)(t) = y^{(n)}(t)$ for $t \in [0,1]$. Consider the boundary value problem

$$1.3) \quad \begin{cases} Ly = f(t, y, \ldots\ldots, y^{(n-1)}), \ t \in [0,1] \\ y \in B \end{cases}$$

where $f(t, P_1, \ldots\ldots, P_n)$ is continuous on $[0,1] \times R^n$, and the associated family of problems

$$(1.3)_\lambda \quad \begin{cases} My = g(t, y, \ldots\ldots, y^{(n-1)}, \lambda), \ 0 \leqslant \lambda \leqslant 1 \\ y \in B \end{cases}$$

where $(My)(t) = \sum_{j=0}^{n} a_j(t)\, y^{(j)}(t)$. Here g, a_j, $j = 0, 1, \ldots, n$ are continuous and $a_n(t) \neq 0$ for $t \in [0,1]$ and also $g(t, v_1, \ldots, v_n, 0) \equiv 0$.

<u>Theorem</u> <u>1.9</u>. Let L, M, f and g be as above. Assume

(i) The problems $(1.3)_\lambda$ and (1.3) are equivalent when $\lambda = 1$ i.e. $(1.3)_\lambda$ and (1.3) have the same set of solutions.

(ii) The differentiable operator (M,B) is invertible as a continuous map from $C^n_B \to C$.

(iii) There is a constant K independent of λ such that $|u|_n < K$ for each solution u to $(1.3)_\lambda$, $0 \leqslant \lambda \leqslant 1$.

Then the boundary value problem (1.3) has at least one solution in $C^n[0,1]$.

Proof Let $\bar{U} = \{u \in C^n_B\ [0,1]: |u|_n \leqslant K\}$ and define $T_\lambda : C^{n-1} \to C$, $0 \leqslant \lambda \leqslant 1$, by $(T_\lambda v)(t) = g(t,v(t),\ldots,v^{(n-1)}(t),\lambda)$. Clearly T_λ is a continuous map. Now $j: C^n_B \to C^{n-1}$ defined by $ju = u$ is the natural embedding. The map j is completely continuous by Corollary 1.8. Then $H_\lambda = M^{-1}\ T_\lambda\ j$ defines a homotopy $H_\lambda : \bar{U} \to C^n_B$. It is clear that the fixed points of H_λ are precisely the solutions to $(1.3)_\lambda$. By (iii) H_λ is fixed point free on ∂U. Moreover, the complete continuity of j together with (ii) and the continuity of T_λ imply that the homotopy H_λ is compact. Hence H_o is homotopic to H_1. Now H_o is essential by Theorem 1.4 since H_o is the zero map. Therefore, Theorem 1.5 implies that H_1 is essential. In particular H_1 has a fixed point, $(1.3)_1$ has a solution and so by (i), (1.3) has a solution.

Remark. If (L,B) is invertible Theorem 1.9 will be applied with $M = L$ and $g = \lambda f$. For example, if B denotes the set of n homogeneous initial condition then (L,B) is invertible by Theorem 1.1. The applications of Theorem 1.9 that occur in the thesis where (L,B) is not invertible fall into one of the following categories:

(i) Take $My=Ly-y$ and

$$g(t,P_1,\ldots,P_n,\lambda) = \lambda[f(t,P_1,\ldots,P_n)-P_1]$$

or

(ii) Take $My=Ly-y'$ and

$$g(t,P_1,\ldots,P_n,\lambda) = \lambda[f(t,P_1,\ldots,P_n)-P_2]$$

The preceding discussion extends to include nonlinear problems in which inhomogeneous boundary conditions occur.

Thus consider the problem

$$\text{1.4)}\qquad \begin{cases} Ly = f(t,y,\ldots,y^{(n-1)}),\ t \in [0,1] \\ V_i(y) = r_i,\ i = 1,\ldots,n \end{cases}$$

where L,f and V_i are as above. We also consider the associated family of problems

$$\text{1.4)}_\lambda\qquad \begin{cases} My = g(t,y,\ldots,y^{(n-1)},\lambda),\ 0 \leqslant \lambda \leqslant 1 \\ V_i(y) = r_i,\ i = 1,\ldots,n \end{cases}$$

where M and g are as above. Let B denote the set of functions y satisfying $V_i(y) = r_i$, $i = 1,\ldots,n$ and B_o the set of functions satisfying the corresponding homogeneous boundary conditions $V_i(y) = 0$, $i = 1,\ldots,n$.

<u>Theorem</u> <u>1.10</u>. Assume

(i) The problems $(1.4)_\lambda$ and (1.4) are equivalent when $\lambda = 1$.

(ii) The differential operator (M,B_o) is invertible as a continuous map from $C^n_{B_o} \to C$.

(iii) There is a constant K independent of λ such that $|u|_n < K$ for each solution u

to $(1.4)_\lambda$, $0 \leqslant \lambda \leqslant 1$.

Then the boundary value problem (1.4) has at least one solution in $C^n[0,1]$.

<u>Proof</u> Define $N:C_B^n \to C$ by $Ny = My$. We will show that $N^{-1}:C \to C_B^n$ exists and is given by $N^{-1} f = M^{-1}f + \ell$ where ℓ is the solution to $My = 0$, $y \in B$. Since (M,B_o) is one to one then N is one to one and it is clear also that $M^{-1}f + \ell$ is the inverse of N provided ℓ exists. We will now show that ℓ exists. Let $u_1,\ldots, u_n$ be n linearly independent solutions of $My = 0$, so that $y = C_1 u_1 +\ldots+ C_n u_n$ is the general solution to $My = 0$. Now since $My = 0$, $y \in B_o$ has only the trivial solution, by (ii), then we must have $\det[V_i(u_j)] \neq 0$. Now to solve $My = 0$, $y \in B$ constants $C_1,\ldots, C_n$ must be found so that $V_i(y) = C_1 V_i(u_1) + C_2 V_i(u_2) +\ldots+ C_n V_i(u_n) = r_i$, $i = 1,\ldots,n$. Since $\det[V_i(u_j)] \neq 0$ there is a unique solution say $k_1,\ldots, k_n$ and $\ell = k_1 u_1+\ldots+k_n u_n$. Thus $N^{-1}:C \to C_B^n$ exists. We now define $\bar{U} = \{y \in C_B^n: |y|_n \leqslant K + |\ell|_n\}$ and $H:\bar{U} \times [0,1] \to C_B^n$ is defined by $H(u,\lambda) = N^{-1} T_\lambda j(u) + (1-\lambda)\ell \equiv H_\lambda$ where $j:C_B^n \to C^{n-1}$ is the completely continuous embedding and $T_\lambda:C^{n-1} \to C$, $0 \leqslant \lambda \leqslant 1$, is the continuous map defined by $(T_\lambda v)(t) = g(t,v(t),\ldots, v^{(n-1)}(t),\lambda)$. This homotopy is clearly compact. Now $H_o = \ell$ and since $|\ell|_n < K + |\ell|_n$ then ℓ is an interior point of $\bar{U}$ in C_B^n. Now Theorem 1.4 implies that H_o is essential. Moreover, $H(u,\lambda) = u$ means $N^{-1} T_\lambda j(u) + (1-\lambda)\ell = u$ which implies $M^{-1} T_\lambda j(u) + \ell +(1-\lambda)\ell = u$. Now since $M\ell = 0$ we have $T_\lambda ju = Mu$. So each fixed point of H satisfies $|u|_n < K$, by (iii), and $H_\lambda(u) = H(u,\lambda)$ is in $K_{\partial U}(\bar{U},C_B^n)$. Hence H_o is homotopic to H_1 and Theorem 1.5 implies that H_1 is essential. Thus H_1 has a fixed point, so $(1.4)_1$, has a solution and so by (i), (1.4) has a solution.

II. First Order Initial Value Problems

2.1 Introduction

The basic existence theorem for the initial value problem

$$\text{2.1)} \qquad \begin{cases} y' = f(t,y) \\ y(0) = r \end{cases}$$

where $f : Z \to R^n$ is continuous and Z is the cylinder $[0,T] \times R^n$, guarantees that a solution exists for $t > 0$ and near 0. Familiar examples show that the interval of existence can be arbitrarily short, depending on the initial value r and the nonlinear behavior of f. In this chapter we see how our analysis which is based on a priori bounds and the topological transversality theorem leads naturally to the study of the dependence of the interval of existence of a solution to (2.1) upon r and f. Additionally the analysis automatically produces best possible results.

2.2 Maximal Intervals of Existence for Classes of Initial Value Problems

The use of a priori bounds to establish existence theorems for boundary value problems is well known. These techniques also apply to initial value problems; however this fact seems to have been largely overlooked. Specializing Theorem 1.9 for initial value problems, we have

Theorem 2.1. Let $f : [0,T] \times R^n \to R^n$ be continuous and $0 \leq \lambda \leq 1$. Suppose there is a constant K independent of λ such that $|y(t)|, |y'(t)| \leq K$ for $t \in [0,T]$

for each solution $y(t)$ to

2.2)$_\lambda$
$$\begin{cases} y' = \lambda f(t,y), \ 0 \leqslant t \leqslant T \\ y(0) = 0. \end{cases}$$

Then the initial value problem

2.2)
$$\begin{cases} y' = f(t,y), \ 0 \leqslant t \leqslant T \\ y(0) = 0 \end{cases}$$

has a solution y in $C^1[0,T]$.

Remark. Theorem 1.9 is formulated for a scaler equation; however the proof extends immediately to the case of systems as in the present formulation.

In view of Theorem 2.1 we obtain immediately

Theorem 2.2. Let $\psi: [0,\infty) \to (0,\infty)$ be continuous, and assume

2.3)
$$|f(t,y)| \leqslant \psi(|y|)$$

for all (t,y). Then the initial value problem (2.2) has a solution in $C^1[0,T]$ for each

2.4)
$$T < T_\infty = \int_0^\infty \frac{du}{\psi(u)} .$$

Moreover, this result is best possible in the sense that the initial value problem

2.5)
$$\begin{cases} y' = \hat{f}(t,y), \ 0 \leqslant t \leqslant \hat{T} \\ y(0) = 0 \end{cases}$$

with $\hat{f}(t,y) = (\psi(|y|), 0,\ldots,0)$ and for which (2.3) holds can have a solution only if $\hat{T} < T_\infty$.

<u>Proof</u> To prove existence of a solution in $C^1[0,T]$ we apply Theorem 2.1. To establish the a priori bounds for $(2.2)_\lambda$, let $y(t)$ be a solution to $(2.2)_\lambda$. Then

$$|y'| = |\lambda f(t,y)| \leqslant \psi(|y|).$$

Now if $|y(t)| \neq 0$ we have by the Cauchy Schwartz Inequality

$$|y|' = \frac{y \cdot y'}{|y|} \leqslant |y'|$$

and the inequality above yields

$$|y|' \leqslant \psi(|y|)$$

at any point t where $y(t) \neq 0$. Suppose $y(t) \neq 0$ for some point $t \in [0,T]$. Since $y(0) = 0$ there is an interval $[a,t]$ in $[0,T]$ such that $|y(s)| > 0$ on $a < s \leqslant t$ and $y(a) = 0$. Then the previous inequality implies

$$\int_a^t \frac{|y(s)|'}{\psi(|y(s)|)}\, ds \leqslant t - a.$$

So, $$\int_0^{|y(t)|} \frac{du}{\psi(u)} \leqslant t - a \leqslant T < T_\infty = \int_0^\infty \frac{du}{\psi(u)}\ .$$

This inequality implies there is a constant M_o such that

$$|y(t)| \leqslant M_0.$$

Now $(2.2)_\lambda$ gives

$$|y'(t)| \leqslant \max_{[0,T]\times[-M_0,M_0]} |f(t,y)| \equiv M_1.$$

So $|y(t)|$, $|y'(t)| \leqslant K = \max\{M_0, M_1\}$ and the existence of a solution to (2.2) is established.

Finally, $y(t) = (y_1(t), \ldots\ldots, y_n(t))$ solves (2.5) if and only if $y_2(t) = \ldots\ldots = y_n(t) = 0$ and $y_1' = \psi(|y_1|)$, $y_1(0) = 0$. Clearly $y_1'(t) > 0$ so $y_1(t) > 0$ and integration yields

$$\int_0^{\hat{T}} \frac{y_1'(s)}{\psi(y_1(s))}\, ds = \hat{T}.$$

Thus, $\hat{T} = \int_0^{y_1(\hat{T})} \frac{du}{\psi(u)} < \int_0^{\infty} \frac{du}{\psi(u)} = T_\infty$ which completes the proof of Theorem 2.2.

<u>Remark</u>. If $T_\infty = \infty$ Theorem 2.2 is called Wintner's Theorem [19], [36].

Theorem 2.1 also holds for the inhomogeneous initial condition $y(0) = r$; see Theorem 1.10. So trivial adjustments in the proof above yield

<u>Theorem</u> <u>2.3</u>. Let $f(t,y)$ and $\psi(y)$ satisfy the hypothesis in Theorem 2.2. Then the initial value problem

$$(2.6)\qquad \begin{cases} y' = f(t,y), & 0 \leqslant t \leqslant T \\ y(0) = r \end{cases}$$

has a solution $y(t)$ in $C^1[0,T]$ for each

$$T < T_\infty = \int_{|r|}^{\infty} \frac{du}{\psi(u)} .$$

Moreover, this result is best possible as described in Theorem 2.2.

2.3 Examples

A few examples illustrate the previous sections results.

<u>Example</u> <u>1.1</u> (Linear and Sublinear Growth)

Suppose $|f(t,y)| \leqslant A(t)\ |y|^p + B(t)$, $0 \leqslant p \leqslant 1$ for bounded functions $A(t)$, $B(t) \geqslant 0$. If A_o and B_o are upper bounds for $A(t)$ and $B(t)$ respectively then

$$|f(t,y)| \leqslant A_o|y|^p + B_o \equiv \psi(|y|)$$

and

$$T_\infty = \int_{|r|}^{\infty} \frac{du}{A_o u^p + B_o} = \infty.$$

Consequently the initial value problem (2.6) has a solution on $[0,T]$ for all $T > 0$.

<u>Example</u> <u>1.2</u> (Polynomial growth)

Suppose $|f(t,y)| \leqslant A(t)\ |y|^m + B(t)$ for $m = 1,2,\ldots.$ and for bounded functions $A(t)$, $B(t) \geqslant 0$. If A_o and B_o are upper bounds for $A(t)$ and $B(t)$, then

$$|f(t,y)| \leq A_o\ |y|^m + B_o \equiv \psi(|y|),\quad T_\infty = \int_{|r|}^{\infty} \frac{du}{A_o u^m + B_o}$$

and the initial value problem (2.6) has a solution on $[0,T]$ for any $T < T_\infty$. In the case of zero initial data, $r = 0$, we have

$$T_\infty = \int_0^\infty \frac{du}{A_o u^m + B_o} = \frac{\pi\ \mathrm{Csc}\ (\pi/m)}{m\ A_o^{1/m}\ B_o^{(m-1)/m}}\ .$$

<u>Example</u> <u>1.3</u> (Estimate of the time before shocks)

The first order quasilinear partial differential equation

$$a(x,y,u)\ u_x + b(x,y,u)\ u_y = c(x,y,u)$$

with suitable assumptions on the coefficients a,b and c can be solved by the method of characteristics. If the solution surface $u = u(x,y)$ is to contain the smooth initial curve

$$x_o = x_o(s),\ y_o = y_o(s),\ u_o = u_o(s)$$

where $0 \leq s \leq 1$ is a parameter, then the characteristic initial value problem is

$$\begin{cases} \frac{dx}{dt} = a,\ \frac{dy}{dt} = b,\ \frac{du}{dt} = c \\ x(0,s) = x_o(s),\ y(0,s) = y_o(s),\ u(0,s) = u_o(s) \end{cases}$$

where $x = x(t,s)$, $y = y(t,s)$, $u = u(t,s)$ and s is regarded as a parameter in the initial value problem. The solution to the initial value problem yields the solution surface by expressing $u(t,s)$ in terms of x

and y after solving $x = x(t,s)$, $y = y(t,s)$ for $t = t(x,y)$ and $s = s(x,y)$. In the region about the initial curve where this can be done, a smooth solution surface results. Suppose we have an estimate on the growth rate of the coefficients in the partial differential equation; say,

$$
\begin{aligned}
|f(x,y,z)| &= |(a(x,y,z),\ b(x,y,z),\ c(x,y,z))| \\
&\leqslant \psi(|(x,y,z)|).
\end{aligned}
$$

Then by Theorem 2.3 no shocks can develop up to the time $T < T_\infty = \int_{|r|}^{\infty} \frac{du}{\psi(u)}$ where

$r = \max_{0 \leqslant s \leqslant 1} |(x_o(s),\ y_o(s),\ u_o(s))|$ assuming of course that t and s are expressible as functions of x and y as required above.

III. Boundary Value Problems on Infinite Intervals

3.1 Introduction

In this chapter we study the existence of solutions to a second order differential equation of the form

$$y'' = f(x,y,y'), \quad 0 \leqslant x < \infty \tag{3.1}$$

where $f(x,u,p)$ is defined and continuous on $[0,\infty) \times R \times R$. We establish that the differential equation (3.1) has bounded solutions under growth conditions of Bernstein type on the nonlinearity f. In addition, existence theorems are established for (3.1) together with the boundary conditions

$$-\alpha y(0) + \beta y'(0) = r \tag{3.2}$$

where $\alpha > 0$, $\beta \geqslant 0$, r is a given constant and

$$\lim_{x\to\infty} y(x) = 0. \tag{3.3}$$

The analysis in this chapter is based on extending theorems of Granas, Guenther and Lee [15], [16], [18] for boundary value problems on finite intervals. As usual $C^k[0,\infty)$ is the space of functions $v(x)$ on $[0,\infty)$ with $v^{(k)}$ continuous, $BC^k[0,\infty)$ is the space of functions $v(x)$ with $v^{(j)}(x)$ bounded and continuous on $[0,\infty)$ for $j = 0,1,\ldots,k$. Let

$$C_o^2[0,\infty) = \{v \in C^2[0,\infty) : \lim_{x\to\infty} v(x) = \lim_{x\to\infty} v'(x) = \lim_{x\to\infty} v''(x) = 0\}$$

with norm $\|v\|_2 = \max\{\|v\|_o, \|v'\|_o, \|v''\|_o\}$ where $\|v\|_o = \sup_{0\leqslant x<\infty} |v(x)|$.

3.2 Global Solutions to Second Order Nonlinear Differential Equations

The Arzela-Ascoli theorem and the following known result about boundary value problems on finite intervals will imply our basic existence theorem for problems defined on $0 \leqslant x < \infty$.

<u>Theorem 3.1.</u> Assume $f(x,u,p)$ is continuous and satisfies

3.4) $\begin{cases} \text{There is a constant } M \geqslant 0 \text{ such that} \\ uf(x,u,0) \geqslant 0 \quad \text{for} \quad |u| > M. \end{cases}$

3.5) $\begin{cases} \text{There are functions } A(x,u),\ B(x,u) > 0 \\ \text{which are bounded when } u \text{ varies in} \\ \text{a bounded set and} \\ |f(x,u,p)| \leqslant A(x,u)\,p^2 + B(x,u). \end{cases}$

Let n be a positive integer and consider the boundary problem

3.6) $\begin{cases} y'' = f(x,y,y'), \quad 0 \leqslant x \leqslant n \\ -\alpha y(0) + \beta y'(0) = r, \quad \alpha > 0 \quad \text{and} \quad \beta \geqslant 0 \\ y(n) = 0. \end{cases}$

Then (3.6) has at least one solution $y_n \in C^2[0,n]$ with $|y_n^{(j)}(t)| \leqslant \tilde{K}$, $j = 0,1,2$ and $t \in [0,n]$, where $\tilde{K}$ is a constant independent of n. In fact let A, B be constants which bound $A(x,u)$ and $B(x,u)$ on $[0,\infty) \times [-M_o, M_o]$ respectively, where $M_o = \max\{M, \frac{|r|}{\alpha}\}$, then $\tilde{K} = \max\{M_o, M_1, AM_1^{\,2} + B\}$ where if $\beta \neq 0$

$$M_1 = \max\left[\frac{|r|+\alpha M_o}{\beta},\ \left[\left(\frac{|r|+\alpha M_o}{\beta}\right)^2 e^{4AM_o} + \frac{B}{A}\left[e^{4AM_o} - 1\right]\right]^{1/2}\right]$$

while if $\beta = 0$

$$M_1 = \max\left[\frac{|r|}{\alpha}, \left[\left(\frac{|r|}{\alpha}\right)^2 e^{4AM_o} + \frac{B}{A}\left(e^{4AM_o}-1\right)\right]^{1/2}\right].$$

The proof follows from a slight modification of arguments in [15], [16] and [17]. These results imply the global soluability of the differential equation (3.1).

Theorem 3.2. Assume $f(x,u,p)$ satisfies (3.4), (3.5). Then the ordinary differential equation

$$y'' = f(x,y,y'), \quad 0 \leqslant x < \infty$$

has at least one solution y in $BC^2[0,\infty)$.

Proof Let n be a positive integer and consider

$$\text{3.7)} \qquad \begin{cases} y'' = f(x,y,y'), & 0 \leqslant x \leqslant n \\ y(0) = 0 \\ y(n) = 0. \end{cases}$$

By Theorem 3.1 there exists a solution $u_n \in C^2[0,n]$ to (3.7) with $|u_n^{(j)}(t)| \leqslant \tilde{K}$, $j = 0,1,2$ and $t \in [0,n]$, where $\tilde{K}$ is a constant independent of n. Now define functions y_n on $[0,\infty)$ by $y_n(x) = u_n(x)$ on $[0,n]$ and $y_n(x) = 0$ on $[n,\infty)$. Clearly each y_n is continuous on $[0,\infty)$ and twice continuously differentiable except possibly at $x = n$. Let $S = \{y_n\}_{n=1}^{\infty}$. By Theorem 1.7 there is a subsequence $\tilde{N}_1$ of the positive integers N and a continuously differentiable function Z_1 on $[0,1]$ such that for $j = 0,1$ $y_n^{(j)}(x) \to Z_1^{(j)}(x)$ uniformly on $[0,1]$ as $n \to \infty$ through $\tilde{N}_1$. Let $N_1 = \tilde{N}_1 | \{1\}$. Then by Theorem 1.7 there is a subsequence $\tilde{N}_2$ of N_1 and a continuously differentiable function Z_2 on $[0,2]$ such that for $j = 0,1$ $y_n^{(j)}(x) \to Z_2^{(j)}(x)$ uniformly on $[0,2]$ as $n \to \infty$ through $\tilde{N}_2$. Note $Z_2 = Z_1$ on $[0,1]$ since $\tilde{N}_2 \subset N_1$. Let $N_2 = \tilde{N}_2 | \{2\}$ and proceed inductively to obtain for $K = 1,2,\ldots.$ a subsequence N_K of positive integers with $N_K \subset N_{K-1}$ and a continuously differentiable function Z_K on $[0,K]$ such that for $j = 0,1$ $y_n^{(j)}(x) \to Z_K^{(j)}(x)$ uniformly on $[0,K]$ as $n \to \infty$ in N_K. Since $N_K \subset N_{K-1}$, $Z_K(x) = Z_{K-1}(x)$ on $[0,K-1]$. Now we define a function y as follows: Fix x in $[0,\infty)$ and let K be a positive integer with $x \leqslant K$. Then define $y(x) = Z_K(x)$. From the construction above, y is well defined. It is clear also that $y \in C^1[0,\infty)$. Now fix x and choose and fix $K \geqslant x$. We have

$$y_n'(x) - y_n'(0) = \int_0^x f(t, y_n(t), y_n'(t))dt$$

for $n \in N_K$. Since $y_n^{(j)}(t)$ converges uniformly to $Z_K^{(j)}(t)$ on $[0,K]$ for $j = 0,1$ let $n \to \infty$ through N_K to find

$$Z_K'(x) - Z_K'(0) = \int_0^x f(t, Z_K(t), Z_K'(t))dt.$$

That is, $$y'(x) - y'(0) = \int_0^x f(t,y(t),y'(t))dt.$$

Consequently $y \in C^2[0,\infty)$ and (3.1) is satisfied. It is clear also that $y(0) = 0$ and $\|y\|_2 \leqslant \tilde{K}$, the constant from Theorem 3.1 and consequently y is in $BC^2[0,\infty)$.

<u>Corollary 3.3</u>. Assume $f(x,u,p)$ satisfies (3.4), (3.5). Then the ordinary differential equation (3.1) with the initial condition (3.2) has at least one solution in $BC^2[0,\infty)$.

<u>Proof</u> This follows directly from the proof of Theorem 3.2 except we consider the boundary value problem

$$\begin{cases} y'' = f(x,y,y'), \ 0 \leqslant x \leqslant n \\ -\alpha y(0) + \beta y'(0) = r, \ \alpha > 0 \quad \text{and} \quad \beta \geqslant 0 \\ y(n) = 0 \end{cases}$$

instead of (3.7).

<u>Remark</u>. Suppose the Sturm-Liouville initial condition (3.2) is replaced by the homogeneous Neumann initial condition

3.8) $$y'(0) = 0.$$

Then it follows easily from results in [15], [16] and the reasoning used above that Corollary 3.3 holds with (3.8) replacing (3.2).

However stronger assumptions on f are needed for existence of a solution to the inhomogeneous Neumann problem on a finite interval; see [18]. Using these results, with (3.2) replaced by

3.9) $$y'(0) = r \neq 0$$

one can prove

<u>Theorem</u> <u>3.4</u>. Assume $f(x,u,p)$ is continuous and satisfies:

3.10) $\begin{cases} \text{There is a constant } K > 0 \text{ such that} \\ f_u(x,u,p) \geqslant K \text{ for } (x,u,p) \text{ in } [0,\infty) \times R \times \{r\}. \end{cases}$

3.11) $\begin{cases} \text{For } 0 \leqslant x < \infty \text{ and } u \in [-|r|,|r|], \ f(x,u,r) \\ \text{is bounded.} \end{cases}$

3.12) $\begin{cases} f \text{ satisfies the Bernstein growth} \\ \text{condition (3.5).} \end{cases}$

Then the ordinary differential equation (3.1) with the initial condition (3.9) has at least one solution in

$BC^2[0,\infty)$.

3.3 Solutions to Second Order Boundary Value Problems on $[0,\infty)$

Suppose f satisfies (3.4), (3.5). Then the reasoning in section 3.2 yields a $BC^2[0,\infty)$ solution to (3.1) and (3.2). Suppose in addition each solution $u \in BC^2[0,\infty)$ of (3.1), (3.2) satisfies the following condition:

3.13) $\begin{cases} \text{There is a continuous function } \psi \text{ such} \\ \text{that } \psi(x) \to 0 \text{ as } x \to \infty \text{ and } |u(x)| \leq \psi(x) \\ \text{for } 0 \leq x < \infty. \end{cases}$

Then a priori each solution to (3.1), (3.2) tends to zero at infinity and we have

<u>Theorem</u> <u>3.5</u>. Assume $f(x,u,p)$ is continuous and (3.4), (3.5) and (3.13) are satisfied. Then the boundary value problem (3.1), (3.2), (3.3) has at least one solutions in $C^2[0,\infty)$.

Bounds of type (3.13) are usually hard to establish. We show how to do this in section 3.4, which establishes that a nonlinear semiconductor problem has a solution. The next results show that (3.13) and the Berstein type growth condition imply similar bounds which guarantee that y' and y'' tend to zero at infinity.

<u>Corollary</u> <u>3.6</u>. Suppose (3.4), (3.5) and (3.13) are satisfied. Then each solution $y \in C^2[0,\infty)$ to (3.1), (3.2), (3.3) satisfies $\lim_{x\to\infty} y'(x) = 0$.

<u>Proof</u> Our strategy is to obtain a bound similar to (3.13) for y'. We can assume without loss of

generality that ψ is a decreasing function with $|y(x)| \leq \psi(x)$, $0 \leq x < \infty$ and $\psi(x) \to 0$ as $x \to \infty$ (Otherwise let $\psi_1(x) = \sup_{t \geq x} \psi(x)$ and apply the following argument with ψ_1 replacing ψ). Since $\lim_{x\to\infty} y(x) = 0$ one can show easily that for each $\epsilon > 0$ and for each $t > 0$ there exists $s > t$ such that $|y'(s)| \leq \epsilon$. Fix $\epsilon > 0$. Let t be a point in $[0,\infty)$ for which $|y'(t)| > \epsilon$. Then there is an interval $[t,s]$ such that y' maintains a fixed sign on $[t,s]$ and $|y'(s)| \leq \epsilon$. Assume $y' \geq 0$ on $[t,s]$. Since $\|y\|_o \leq \|\psi\|_o$ by (3.13), there are constants $A,B > 0$ which bound $A(x,u)$ and $B(x,u)$ on $[0,\infty) \times [-\|\psi\|_o, \|\psi\|_o]$. Consequently, the differential equation and (3.5) yield

$$-2\,A\,y' \leq \frac{2Ay'y''}{Ay'^2+B}$$

and integrating from t to s yields

$$\mathrm{Ln}\left[\frac{A\;y'^2(t)+B}{A\;y'^2(s)+B}\right] \leq 2\,A[\psi(t) + \psi(s)] \leq 4\,A\,\psi(t)$$

because ψ is decreasing. Hence

$$|y'(t)| \leq \left[\epsilon^2\; e^{4A\psi(t)} + \frac{B}{A}\left[e^{4A\psi(t)} - 1\right]\right]^{1/2}.$$

The same bound is obtained if we assume $y' \leq 0$ on $[t,s]$ and this inequality holds trivially if $|y'(t)| \leq \epsilon$. Letting $\epsilon \to 0^+$ yields

$$3.14) \qquad |y'(t)| \leq \left[\frac{B}{A}\left[e^{4A\psi(t)} - 1\right]\right]^{1/2} \equiv \psi_2(t)$$

for all $0 \leq t < \infty$. Clearly $\psi_2(t)$ is continuous on $[0,\infty)$ and $\psi_2(x) \to 0$ as $x \to \infty$.

The main result in this section is concerned with obtaining $C_o^2[0,\infty)$ solutions to the boundary value problem (3.1), (3.2) and (3.3).

Theorem 3.7. Assume (3.4), (3.5), (3.13) are satisfied. Suppose in addition $f(x,u,p)$ satisfies:

3.15) $$\lim_{x\to\infty} f(x,0,0) = 0$$

3.16) $\begin{cases}\text{For } 0 \leqslant x < \infty \text{ and } u,p \text{ each in a bounded} \\ \text{interval } f_u(x,u,p) \text{ and } f_p(x,u,p) \text{ are bounded.}\end{cases}$

Then the boundary value problem (3.1), (3.2), (3.3) has at least one solution in $C_o^2[0,\infty)$.

Proof In view of Theorem 3.5 and Corollary 3.6 there exists a solution $y \in BC^2[0,\infty)$ to (3.1), (3.2) which satisfies $\lim_{x\to\infty} y(x) = \lim_{x\to\infty} y'(x) = 0$. The differential equation yields

$$\begin{aligned} y''(x) &= f(x,y,y') - f(x,0,0) + f(x,0,0) \\ &= \bar{f}_y\, y + \bar{f}_{y'}\, y' + f(x,0,0) \end{aligned}$$

where $\bar{f}_y$, $\bar{f}_{y'}$ are intermediate values obtained by the Mean Value Theorem. Now (3.13), (3.4) and (3.16) imply

$$|y''(x)| \leqslant S\,\psi(x) + R\,\psi_2(x) + f(x,0,0) \equiv \psi_3(x)$$

where $S = \sup|f_u(x,u,p)|$, $R = \sup|f_p(x,u,p)|$ for $(x,u,p) \in [0,\infty) \times [-K,K] \times [-K,K]$, where $K \geqslant \|y\|_1$ is a constant. Clearly $\psi_3(x)$ is continuous on $[0,\infty)$ and $\psi_3(x) \to 0$ as $x \to \infty$ by (3.15).

3.4 Applications to Semiconductor Devices

In studying the theory of semiconductor devices one is led to boundary value problems for the Poisson equation in unbounded domains which can consist of several layers. In one spacial dimension the problem reduces to finding solutions $u = u(x)$ and $v = v(x)$ to

$$\begin{aligned}
&v'' = 0, \ -1 < x < 0 \\
&v(-1) = \frac{\phi}{\alpha} \\
&u'' = f(x,u), \ 0 < x < \infty \\
&v(0) = u(0) \\
&\alpha\, v'(0) = u'(0) \\
&\lim_{x\to\infty} u(x) = 0
\end{aligned}$$

where $\alpha > 0$ and ϕ are given constants. In semiconductor applications, the function f is usually written in the form

$$f(x,u) = -\frac{q}{\epsilon}\left[p-n + N_D - N_A + N_Q\right]$$

where n and p refer to the electron and hole mobile charge densities, N_D and N_A the donor and acceptor impurity densities and N_Q is the density of the fixed charged particles. The constant q is the magnitude of the electronic charge, and ϵ is the dielectric permitivity. See [5], [6], [32] for an in-depth physical discussion. Some common examples which are used for the function f are:

$$\begin{aligned}
&f(x,u) = Ku \\
&f(x,u) = A \sinh (Ku) \\
&f(x,u) = -A - B\, e^{-cu} + (A+B)e^{cu} + S\, e^{-\sigma(x-x_m)^2}
\end{aligned}$$

Here A, B, S, c, K, σ are all positive constants and $x_m > 0$ is a fixed value.

Since v is a linear function on $-1 \leqslant x \leqslant 0$ it can be eliminated and thus the following boundary value problem for u is obtained

3.17) $$u'' = f(x,u)$$

3.18) $$u'(0) - \alpha\, u(0) = -\phi$$

3.19) $$\lim_{x \to \infty} u(x) = 0.$$

Motivated by the examples cited above, we shall make the following assumptions on the function $f(x,u)$:

3.20) $$\begin{cases} f(x,u) \text{ is continuous on } [0,\infty) \times R. \\ \text{Moreover for } 0 \leqslant x < \infty \text{ and } u \text{ in a bounded} \\ \text{interval } f(x,u),\ f_u(x,u) \text{ are bounded.} \end{cases}$$

3.21) $$\begin{cases} \text{There is a constant } m > 0 \text{ such that} \\ \frac{\partial f}{\partial u}(x,u) \geqslant m^2 \text{ on } [0,\infty) \times R. \end{cases}$$

3.22) $$\lim_{x \to \infty} f(x,0) = 0.$$

Uniqueness for the boundary value problem (3.17), (3.18), (3.19) follows from an elementary maximum principle and (3.21).

<u>Lemma 3.8</u>. Assume $f(x,u)$ satisfies (3.21). Then the boundary value problem (3.17), (3.18), (3.19) has at most one solution.

<u>Proof</u> Suppose $u_1(x)$ and $u_2(x)$ are solutions to

(3.17), (3.18), (3.19). The function $\frac{(u_1-u_2)^2}{2}$ vanishes at infinity; so if it is not identically zero, it must have a positive maximum at a point $\tilde{x}$, $0 \leqslant \tilde{x} < \infty$. Now if $\tilde{x} > 0$, we would have

$$0 \geqslant \frac{d^2}{dx^2} \frac{(u_1-u_2)^2}{2}\bigg|_{x=\tilde{x}} = \left[\tilde{u}_1 - \tilde{u}_2\right]\left[f(\tilde{x},\tilde{u}_1) - f(\tilde{x},\tilde{u}_2)\right]$$

i.e. $\qquad 0 \geqslant m^2(\tilde{u}_1 - \tilde{u}_2)^2$

where $\tilde{u}_1 = u_1(\tilde{x})$ and $\tilde{u}_2 = u_2(\tilde{x})$. Since $(\tilde{u}_1-\tilde{u}_2)^2 > 0$ this is a contradiction. If on the other hand, $\tilde{x} = 0$ then the boundary condition at zero gives

$$\frac{d}{dx} \frac{(u_1-u_2)^2}{2}\bigg|_{x=0} = \alpha(\tilde{u}_1 - \tilde{u}_2)^2 > 0$$

which contradicts the fact that $\frac{(u_1-u_2)^2}{2}$ has its nonzero maximum at $\tilde{x} = 0$. These contradictions show that $\frac{(u_1-u_2)^2}{2} \equiv 0$.

The existence of a solution to (3.17), (3.18), (3.19) will follow from the results of section 3.2 and 3.3. Our analysis also uses the fact that a special linear problem of type (3.17), (3.18), (3.19) can be solved explicitly.

<u>Proposition 3.9</u>. Let $m > 0$, β real and $h(x)$ be a continuous function on $[0,\infty)$ with limit zero at infinity. Then the boundary value problem

$$\text{3.23)} \qquad \begin{cases} w'' - m^2 w = h(x), \; 0 \leqslant x < \infty \\ w'(0) - \alpha w(0) = \beta \\ \lim\limits_{x\to\infty} w(x) = 0 \end{cases}$$

has a unique solution. Moreover, if $h(x) \leqslant 0$ and $\beta \leqslant 0$ then $w(x)$ is a nonnegative function on $[0,\infty)$. Consequently if $h(x) \geqslant 0$ and $\beta \geqslant 0$ then $w(x)$ is a nonpositive function on $[0,\infty)$.

Proof The existence and uniqueness of the solution to (3.23) can be confirmed by quadrature. In fact we can obtain via variation of parameters that the solution is

$$w(x) = -\left[\frac{\beta + \int_0^\infty e^{-mt} h(t)dt}{\alpha + m}\right] e^{-mx}$$

$$- \int_0^x e^{-m(x-\zeta)} \left[\int_\zeta^\infty e^{-m(t-\zeta)} h(t)dt\right] d\zeta .$$

Evidently $w(x) \geqslant 0$ when $\beta \leqslant 0$ and $h(x) \leqslant 0$.

Theorem 3.10. Assume $f(x,u)$ satisfies (3.20), (3.21), (3.22). Then the boundary value problem (3.17), (3.18), (3.19) has at least one solution in $C_o^2[0,\infty)$.

Proof We begin by showing $f(x,u)$ satisfies (3.4), (3.5). We have

$$\begin{aligned} uf(x,u) &= u[f(x,u) - f(x,0)] + uf(x,0) \\ &\geqslant m^2 u^2 + uf(x,0) \end{aligned}$$

by the mean value theorem and (3.21). Hence

$$uf(x,u) > 0 \quad \text{for} \quad |u| > M_3 \quad \text{where} \quad M_3 = \frac{\max\limits_{0 \leqslant x < \infty} |f(x,0)|}{m^2} .$$

So (3.4) is satisfied. Now (3.20) clearly implies

(3.5). Thus Corollary 3.3 implies there exists $u \in BC^2[0,\infty)$ which satisfies (3.17), (3.18). Since $u \in BC^2[0,\infty)$ there exists a constant K such that $\|u\|_2 \leq K$. Next we show $\lim_{x\to\infty} u(x) = 0$. Let $w_1(x)$ be the unique nonnegative solution to

$$\begin{cases} w'' - m^2 w = f(x,0) - |f(x,0)|, \quad 0 \leq x < \infty \\ w'(0) - \alpha\, w(0) = -\phi - |\phi| \\ \lim_{x\to\infty} w(x) = 0 \end{cases}$$

guaranteed by Proposition 3.9. Now consider the function $r(x) = u(x) - w_1(x)$. We shall show it is never positive. First, we show $r(x)$ cannot have a local positive maximum on $[0,\infty)$. Indeed, suppose a local positive maximum were to occur at $\tilde{x}$. If $\tilde{x} > 0$ we would have

$$0 \geq \frac{d^2}{dx^2}\,(u(x) - w_1(x))\Big|_{x=\tilde{x}} = u''(\tilde{x}) - w_1''(\tilde{x})$$

$$= f(\tilde{x},u(\tilde{x})) - m^2 w_1 - f(\tilde{x},0) + |f(\tilde{x},0)|$$

$$= \bar{f}_u(u(\tilde{x}) - w_1(\tilde{x})) + (\bar{f}_u - m^2)\, w_1(\tilde{x}) + |f(\tilde{x},0)|$$

$$> 0$$

where $\bar{f}_u$ is an intermediate value obtained from the mean value theorem, and this is a contradiction. On the other hand, if $\tilde{x} = 0$ we would have

$$0 \geq \frac{d}{dx}\,(u(x) - w_1(x))\Big|_{x=0} = u'(0) - w_1'(0)$$

$$= \alpha\, u(0) - \phi - \alpha\, w_1(0) + \phi + |\phi|$$

$$= \alpha\,(u(0) - w_1(0)) + |\phi|$$

$$> 0$$

which is also impossible. Consequently, as asserted,

$r(x)$ cannot have a positive local maximum on $[0,\infty)$. It now follows easily that $r(x) \leqslant 0$ on $[0,\infty)$: If not, then there is a $c \geqslant 0$ such that $r(c) > 0$, and since $r(x)$ cannot have a positive local maximum on $[0,\infty)$ it follows that $r(x_2) > r(x_1)$ for all $x_2 > x_1 \geqslant c$; otherwise $r(x)$ would have a local positive maximum on $[0,x_2]$. Thus $r(x) = u(x) - w_1(x)$ is strictly increasing for $x \geqslant c$. Since both $u(x)$ and $w_1(x)$ are bounded on $[0,\infty)$ and $w_1(x) \to 0$ as $x \to \infty$, it follows that there exists

$$\lim_{x\to\infty} u(x) = \lim_{x\to\infty} (u(x) - w_1(x)) = \ell > 0.$$

Then the relation

$$\begin{aligned} u'' &= f(x,u(x)) - f(x,0) + f(x,0) \\ &= \bar{f}_u\, u(x) + f(x,0) \geqslant m^2\, u(x) + f(x,0) \end{aligned}$$

and the fact that $f(x,0) \to 0$ as $x \to \infty$ implies that there exists $c_1 \geqslant c$ such that

$$u''(x) \geqslant \frac{m^2\ell}{2} \quad \text{for all} \quad x \geqslant c_1.$$

This contradicts the boundedness of $u(x)$ on $[0,\infty)$, and completes the proof that $u(x) - w_1(x) \leqslant 0$. Similar reasoning establishes that $u(x) - w_2(x) \geqslant 0$ for all $0 \leqslant x < \infty$ where $w_2(x)$ is the unique nonpositive solution to

$$\begin{cases} w''-m^2\, w = |f(x,0)| + f(x,0), & 0 \leqslant x < \infty \\ w'(0) - \alpha w(0) = -\phi + |\phi| \\ \lim\limits_{x\to\infty} w(x) = 0. \end{cases}$$

Consequently $|u(x)| \leqslant \psi(x) = \max(w_1(x), -w_2(x))$ for $0 \leqslant x < \infty$. Thus, the hypothesis of Theorem 3.7 are satisfied and the solution $u(x)$ to (3.17), (3.18), (3.19) belongs to $C_o^2[0,\infty)$.

IV. Ordinary Differential Equations In The Complex Domain

4.1 Introduction

Ordinary differential equations in the complex domain have been studied in great depth this century; however, very little is known on intervals of existence of a solution. We are interested in this chapter in applying the ideas of Chapter I to study the existence of solutions to the initial value problem in the complex domain:

$$\text{4.1)}\qquad \begin{cases} y^{(n)} = f(t,y,y',\ldots,y^{(n-1)}), \ t \in U_T \\ y(0) = \alpha_o,\ y'(0) = \alpha_1,\ldots,\ y^{(n-1)}(0) = \alpha_{n-1} \end{cases}$$

where $\alpha_j \in \mathbb{C}$, $j = 0,1,\ldots,n-1$ and $f:\bar{U}_T \times \mathbb{C}^n \to \mathbb{C}$ is analytic in $t,y,\ldots,y^{(n-1)}$ for $t \in U_T$ and continuous in $t,y,\ldots y^{(n-1)}$ for $t \in \bar{U}_T$. Here of course $\mathbb{C}$ is the complex plane and $U_T = \{z:|z| < T\}$, $T \in R$.

By a solution to (4.1) we mean a function y which is analytic on U_T with $y,y',\ldots,y^{(n)}$ continuous on $\bar{U}_T$ which satisfies the differential equation and initial conditions. The basic existence theorems in Ince [20] and Smart [31] guarantees that a solution exists for $|t| < \epsilon$ for some $\epsilon > 0$ suitably small; however, from these theorems it is extremely difficult and many times impossible to produce a specific interval of existence of a solution. The results of this chapter enables us to read off immediately from the differential equation an interval of existence of a solution, although it may not be maximal. We in fact establish with a restriction on T (which depends only on the initial data and the growth constants associated with the nonlinear behavior of f)

that the initial value problem (4.1) has bounded solutions $y(t)$ for $t \in \bar{U}_T$ under growth conditions of Bernstein type on the nonlinearity f. Furthermore, in section 4.4 we obtain existence theorems for "Boundary Value Problems" in the complex domain.

Let $A^k(\bar{U}_T)$, $k \geqslant 0$ an integer, denote the Banach space of functions $g, g: \bar{U}_T \to \mathbb{C}$ is analytic on U_T with $g, g', \ldots, g^{(k)}$ continuous on $\bar{U}_T$ with norm

$$\|g\|_k = \max\{\|g\|_o, \ldots, \|g^{(k)}\|_o\}$$

where $\|g\|_o = \sup\limits_{t \in \bar{U}_T} |g(t)|$.

If B denotes the initial conditions in (4.1) then let $A_B^k(\bar{U}_T)$ denote the subset of functions in $A^k(\bar{U}_T)$ which satisfy the given initial conditions. For notational purposes we also let $A^o(\bar{U}_T) = A(\bar{U}_T)$.

Remark. Let $H^k(U_T)$ denote the space of functions $g, g: U_T \to \mathbb{C}$ is analytic on U_T and $g^{(j)}$, $j = 0, \ldots, k$ is bounded on U_T with norm

$$\|g\|_k = \max\{\|g\|_o, \ldots, \|g^{(k)}\|_o\}$$

where $\|g\|_o = \sup\limits_{t \in U_T} |g(t)|$.

It is more natural to seek solutions $y \in H^2(U_T)$ to (4.1). However, the a priori bounds derived in the forthcoming sections together with the Mean Value Theorem for complex functions (Proposition 1.17 in Conway [7]), Theorem 9.1 of McShane and Botts [4] and the assumptions on f imply, a priori, that $y \in A^k(\bar{U}_T)$.

4.2 Solutions to First Order Initial Value Problems in the Complex Domain

We begin by extending Theorem [1.9] for the new class of problems.

<u>Theorem</u> <u>4.1</u>. Let $f:\bar{U}_T \times \mathbb{C} \to \mathbb{C}$ be analytic in $t, y, \ldots, y^{(n-1)}$ for $t \in U_T$ and continuous in $t, y, \ldots y^{(n-1)}$ for $t \in \bar{U}_T$. Suppose there is a constant K such that $|y(t)|$, $|y'(t)| \leqslant K$ for $t \in \bar{U}_T$ for each solution $y(t)$ to

$$4.2)_\lambda \qquad \begin{cases} y' = \lambda\, f(t,y), \quad 0 \leqslant \lambda \leqslant 1 \\ y(0) = 0. \end{cases}$$

Then the initial value problem

$$4.2) \qquad \begin{cases} y' = f(t,y), \quad t \in U_T \\ y(0) = 0 \end{cases}$$

has a solution y in $A^1(\bar{U}_T)$.

<u>Proof</u> Let $\bar{V} = \{u \in A^1_B(\bar{U}_T) : \|u\|_1 \leqslant K + 1\}$, $F_\lambda : A(\bar{U}_T) \to A(\bar{U}_T)$, $0 \leqslant \lambda \leqslant 1$, be the continuous map defined by $(F_\lambda u)(t) = \lambda f(t, v(t))$, and $j : A^1_B(\bar{U}_T) \to A(\bar{U}_T)$ be the natural embedding which is completely continuous by Theorem 1.7. We define $L : A^1_B(\bar{U}_T) \to A(\bar{U}_T)$ by $Ly = y'$. It follows from Theorem 1.6 that L^{-1} is a bounded linear operator. Then $H_\lambda = L^{-1} F_\lambda\, j$ defines a homotopy $H_\lambda : \bar{V} \to A^1_B(\bar{U}_T)$. It is clear that the fixed points of H_λ are precisely the solutions to $(4.2)_\lambda$ and hence H_λ is fixed point free on ∂V. Moreover, the complete continuity of j together with the continuity of L^{-1} and F_λ imply that the homotopy H_λ is compact. H_o is essential by Theorem 1.4 and Theorem 1.5 implies that H_1 is essential. Thus (4.2) has a solution.

We establish various existence theorems by placing growth conditions on the nonlinearity f. Suppose at first f satisfies linear or sublinear growth conditions, then Theorem 4.1 and the maximum modulus principle yield

<u>Theorem</u> <u>4.2</u>. (Linear growth)
Suppose

$$|f(t,u)| \leqslant A(t)\ |u| + B(t) \tag{4.3}$$

where $A(t)$, $B(t) \geqslant 0$ are functions bounded on bounded t-sets. Let A and B denote the upper bounds for $A(t)$, $B(t)$ respectively for $t \in \bar{U}_T$. Then (4.2) has at least one solution in $A^1(\bar{U}_T)$ provided $0 < T < 1/A$.

<u>Proof</u> Fix T with $0 < T < \frac{1}{A}$. The existence of a solution in $A^1(\bar{U}_T)$ will follow immediately from Theorem 4.1 once we establish a priori bounds for $(4.2)_\lambda$. Let $y \in A^1(\bar{U}_T)$ be a solution to $(4.2)_\lambda$. Then

$$|y'| = |\lambda f(t,y)| \leqslant A|y| + B.$$

Suppose the maximum of $|y(t)|$ for $t \in \bar{U}_T$ occurs at ζ. If $\zeta \in U_T$ then by the maximum modulus principle $y \equiv$ constant. The initial condition yields $y \equiv 0$. On the other hand if $\zeta \in \partial U_T$ then

$$|y'| \leqslant A|y(\zeta)| + B$$

and integration along the straight line from 0 to ζ yields

$$|y(\zeta)| \leqslant AT|y(\zeta)| + BT.$$

Hence

$$|y(\zeta)| \leq \frac{BT}{1-AT} \equiv M_o, \quad \text{because} \quad AT < 1.$$

This inequality implies

$$|y(t)| \leq M_o \quad \text{for} \quad t \in \bar{U}_T.$$

Now (4.3) gives

$$|y'(t)| \leq AM_o + B \equiv M_1.$$

Thus $|y(t)|$, $|y'(t)| \leq K = \max\{M_o, M_1\}$ and the existence of a solution to (4.2) is established.

The main result of this section is concerned with obtaining solutions to the initial value problem (4.2) with Bernstein growth conditions on the nonlinearity f. Here the continuity of the maximum modulus function together with Theorem 4.1 yields

Theorem 4.3. (Bernstein growth)
Suppose

4.4) $$|f(t,u)| \leq A(t)\ |u|^2 + B(t)$$

where $A(t)$, $B(t) \geq 0$ are functions bounded on bounded t-sets. Let A and B denote the upper bounds for $A(t)$, $B(t)$ respectively for $t \in \bar{U}_T$. Then (4.2) has at least one solution in $A^1(\bar{U}_T)$ provided $0 < T < \frac{1}{2(AB)^{1/2}}$.

Proof Fix T with $0 < T < \frac{1}{2(AB)^{1/2}}$. The existence of a solution will follow immediately from Theorem 4.1 once we establish a priori bounds for $(4.2)_\lambda$. Let $y \in A^1(\bar{U}_T)$ be a solution to $(4.2)_\lambda$. Then

$$|y'| = |\lambda f(t,y)| \leq A|y|^2 + B.$$

Suppose the maximum of $|y(t)|$ for $t \in \bar{U}_T$ occurs at ζ. If $\zeta \in U_T$ then $y \equiv 0$. Otherwise $\zeta \in \partial U_T$ and so

$$|y(\zeta)| \leq AT|y(\zeta)|^2 + BT.$$

Hence this quadratic inequality implies that either

$$|y(\zeta)| \leq \frac{1-(1-4ABT^2)^{1/2}}{2AT} \quad \text{or} \quad |y(\zeta)| \geq \frac{1+(1-4ABT^2)^{1/2}}{2AT}.$$

Apply the same reasoning for any r with $0 < r \leq T$ to obtain either

$$|y(t)| \leq \frac{1-(1-4ABr^2)^{1/2}}{2Ar} \quad \text{or} \quad |y(t)| \geq \frac{1+(1-4ABr^2)^{1/2}}{2Ar}$$

for $t \in \bar{U}_r$. Since $y(0)=0$ and $\lim_{r\to 0^+} \frac{1+(1-4ABr^2)^{1/2}}{2Ar} = \infty$

there exists an r, $0 < r \leq T$ such that

$$|y(t)| \leq \frac{1-(1-4ABr^2)^{1/2}}{2Ar} \quad \text{for} \quad t \in \bar{U}_r.$$

In particular, since $h(r) = \frac{1-(1-4ABr^2)^{1/2}}{2Ar}$ is an increasing function on $\left[0, \frac{1}{2(AB)^{1/2}}\right]$, we have

$$M(r) = \max_{t\in\bar{U}_r} |y(t)| \leq \frac{1-(1-4ABr^2)^{1/2}}{2Ar} < \left(\frac{B}{A}\right)^{1/2}.$$

Suppose $\bar{U}_r$ is the largest closed disc where

$M(r) \leq \frac{1-(1-4ABr^2)^{1/2}}{2Ar}$. If $r = T$ then we have

$|y(t)| \leq \frac{1-(1-4ABT^2)^{1/2}}{2AT}$ for $t \in \bar{U}_T$. If on the other hand $r < \eta < T$ then

$$M(\eta) \geq \frac{1+(1-4AB\eta^2)^{1/2}}{2A\eta} \geq \frac{1}{2A\eta} > \left(\frac{B}{A}\right)^{1/2}$$

because of $0 < T < \frac{1}{2(AB)^{1/2}}$. Since $M(p)$ is a

continuous function of p for $0 \leq p \leq T$ then there

exists a μ, $r < \mu < \eta$ such that $M(\mu) = \left(\frac{B}{A}\right)^{1/2}$.
Since

$$M(\mu) \geqslant \frac{1+(1-4AB\mu^2)^{1/2}}{2A\mu} > \left(\frac{B}{A}\right)^{1/2}$$

we have a contradiction. Hence

$$|y(t)| \leqslant \frac{1-(1-4ABT^2)^{1/2}}{2AT} \equiv M_o \quad \text{for} \quad t \in \bar{U}_T.$$

Now (4.4) gives

$$|y'(t)| \leqslant AM_o^2 + B \equiv M_1.$$

Thus $|y(t)|$, $|y'(t)| \leqslant K = \max\{M_o, M_1\}$ and the existence of a solution to (4.2) is established.

Theorem 4.1 also holds for the inhomogeneous initial condition $y(0) = \alpha$, $\alpha \in \mathbb{C}$, via Theorem 1.10. So trivial adjustments in the proofs above yield the following two theorems.

<u>Theorem</u> <u>4.4</u>. Suppose

$$|f(t,u)| \leqslant A(t)|u| + B(t)$$

where $A(t)$, $B(t) \geqslant 0$ are functions bounded on bounded t-sets. Let A and B denote the upper bounds for $A(t)$, $B(t)$ respectively for $t \in \bar{U}_T$. Then

$$\text{4.5)} \qquad \begin{cases} y' = f(t,y), & t \in U_T \\ y(0) = \alpha \end{cases}$$

has at least one solution in $A^1(\bar{U}_T)$ provided $0 < T < 1/A$.

<u>Theorem</u> <u>4.5</u>. Suppose

$$|f(t,u)| \leqslant A(t)|u|^2 + B(t)$$

where $A(t), B(t) \geq 0$ are functions bounded on bounded t-sets. Let A and B denote the upper bounds for $A(t)$, $B(t)$ respectively for $t \in \bar{U}_T$. Then (4.5) has at least one solution in $A^1(\bar{U}_T)$ provided

$$0 < T < \frac{-|\alpha|+(|\alpha|^2+B/A)^{1/2}}{2B}.$$

Remark. We can extend the proof of Theorem's 4.1, 4.2, 4.3, 4.4, 4.5 to the case of first order systems as in their present formulation. Thus we obtain for example:

Theorem 4.6. Suppose for $f:\bar{U}_T \times \mathbb{C}^n \to \mathbb{C}^n$ we have

$$|f(t,u)| \leq A(t)\,|u|^2 + B(t)$$

where $A(t), B(t) \geq 0$ are functions bounded on bounded t-sets. Let A and B denote the upper bounds for $A(t)$, $B(t)$ respectively for $t \in \bar{U}_T$. Then

$$\begin{cases} y' = f(t,y), & t \in U_T \\ y(0) = \alpha, & \alpha \in \mathbb{C}^n \end{cases}$$

has at least one solution in $A^1(\bar{U}_T)$ provided

$$0 < T < \frac{-|\alpha|+(|\alpha|^2+B/A)^{1/2}}{2B}.$$

4.3 Solutions to Higher Order Initial Value Problems in the Complex Domain

Theorem 4.6 produces an interval of existence for solutions to higher order initial value problems; however, the results of this section yields better intervals of existence for such problems. Also some analysis is needed to produce an interval of existence from Theorem 4.6 whereas the conclusions of this section enables us to write down immediately an interval of existence simply by looking at the ordinary differential equation. Furthermore, the ideas of this section indicate a very natural way of looking at "Boundary Value Problems" in the complex domain. Trivial adjustments in the proof of Theorem 1.10 and Theorem 4.1 yield

<u>Theorem</u> <u>4.7</u>. Let $f:\bar{U}_T \times \mathbb{C}^n \to \mathbb{C}$ be analytic in $t,y,\ldots\ldots,y^{(n-1)}$ for $t \in U_T$ and continuous in $t,y,\ldots\ldots,y^{(n-1)}$ for $t \in \bar{U}_T$. Suppose there is a constant K such that $|y(t)|,\ldots\ldots,|y^{(n)}(t)| \leqslant K$ for $t \in \bar{U}_T$ for each solution $y(t)$ to

$$4.6)_\lambda \qquad \begin{cases} y^{(n)} = \lambda f(t,y,y',\ldots\ldots,y^{(n-1)}), \ 0 \leqslant \lambda \leqslant 1 \\ y(0) = \alpha_o, \ y'(0) = \alpha_1,\ldots\ldots,y^{(n-1)}(0) = \alpha_{n-1} \end{cases}$$

where $\alpha_i \in \mathbb{C}$ for $i = 0,1,\ldots\ldots,n-1$. Then the initial value problem

$$4.6) \qquad \begin{cases} y^{(n)} = f(t,y,y',\ldots\ldots,y^{(n-1)}), \ t \in U_T \\ y(0) = \alpha_o,\ldots\ldots\ldots\ldots,y^{(n-1)}(0) = \alpha_{n-1} \end{cases}$$

has a solution y in $A^n(\bar{U}_T)$.

We will formulate the theorems in this section for the homogeneous case $n = 2$. So for the remainder of this section we assume $\alpha_o = \alpha_1 = 0$ and $n = 2$. Now Theorem 4.7 together with the maximum modulus principle will yield

<u>Theorem</u> <u>4.8</u>. (Linear growth)
Suppose

$$4.7) \qquad |f(t,u,p)| \leqslant A(t)|u| + B(t)|p| + C(t)$$

where $A(t)$, $B(t)$, $C(t) \geqslant 0$ are functions bounded on bounded t-sets. Let A, B and C denote the upper bounds for $A(t)$, $B(t)$, $C(t)$ respectively for $t \in \bar{U}_T$. Then (4.6) has at least one solution in $A^2(\bar{U}_T)$ for

$0 < T < \frac{-B+(B^2+4A)^{1/2}}{2A}$.

<u>Proof</u> Fix T with $0 < T < \frac{-B+(B^2+4A)^{1/2}}{2A}$. We obtain first a priori bounds for $(4.6)_\lambda$. Let $y \in A^2(\bar{U}_T)$ be a solution to $(4.6)_\lambda$. Then

$$|y''| \leqslant A\,|y| + B|y'| + C.$$

Suppose the maximum of $|y'(t)|$ for $t \in \bar{U}_T$ occurs at ζ. If $\zeta \in U_T$ then the maximum modulus principle and initial conditions yield $y \equiv 0$. On the other hand if $\zeta \in \partial U_T$ then

4.8) $$|y(t)| \leqslant T|y'(\zeta)| \quad \text{for any} \quad t \in \bar{U}_T.$$

Thus

$$|y''(t)| \leqslant AT|y'(\zeta)| + B|y'(\zeta)| + C$$

and integrating along the straight line from 0 to ζ yields

$$|y'(\zeta)| \leqslant \frac{CT}{1-[AT^2+BT]} \equiv M_o,$$

because $0 < T < \frac{-B+(B^2+4A)^{1/2}}{2A}$. Hence $|y'(t)| \leqslant M_o$ for $t \in \bar{U}_T$ and so (4.8) implies $|y(t)| \leqslant TM_o \equiv M_1$ for $t \in \bar{U}_T$. Now (4.7) gives

$$|y''(t)| \leqslant AM_1 + BM_o + C \equiv M_2.$$

Thus $|y(t)|$, $|y'(t)|$, $|y''(t)| \leqslant K = \max\{M_o, M_1, M_2\}$ and the existence of a solution to (4.6) follows from Theorem 4.7.

Finally, suppose in place of (4.7) f satisfies Bernstein type growth conditions then Theorem 4.7 and the continuity of the maximum modulus function $M(p)$, $0 \leq p \leq T$, will yield our main result.

<u>Theorem 4.9</u>. (Bernstein growth)
Suppose

$$4.9) \qquad |f(t,u,p)| \leq A(t)\,|u|^2 + B(t)\,|p|^2 + C(t)$$

where $A(t)$, $B(t)$, $C(t) \geq 0$ are functions bounded on bounded t-sets. Let A, B and C denote the upper bounds for $A(t)$, $B(t)$, $C(t)$ respectively for $t \in \bar{U}_T$. Then (4.6) has at least one solution in $A^2(\bar{U}_T)$ provided $0 < T < \left[\frac{-BC+(B^2C^2+AC)^{1/2}}{2AC}\right]^{1/2}$.

<u>Proof</u> Fix T with $0 < T < \left[\frac{-BC+(B^2C^2+AC)^{1/2}}{2AC}\right]^{1/2}$. Let $y \in A^2(\bar{U}_T)$ be a solution to $(4.6)_\lambda$ and suppose the maximum of $|y'(t)|$ for $t \in \bar{U}_T$ occurs at ζ. If $\zeta \in U_T$ then $y \equiv 0$. Otherwise $\zeta \in \partial U_T$ and so

$$|y'(\zeta)| \leq AT^3|y'(\zeta)|^2 + BT\,|y'(\zeta)|^2 + CT.$$

Hence this quadratic inequality implies that either

$$|y'(\zeta)| \leq \frac{1-(1-4(AT^3+BT)CT)^{1/2}}{2(AT^3+BT)}$$

or

$$|y'(\zeta)| \geq \frac{1+(1-4(AT^3+BT)CT)^{1/2}}{2(AT^3+BT)}.$$

Apply the same reasoning for any r with $0 < r \leq T$ to obtain either

$$|y'(t)| \leq \frac{1-(1-4(Ar^3+Br)Cr)^{1/2}}{2(Ar^3+Br)}$$

or $\quad |y'(t)| \geq \dfrac{1+(1-4(Ar^3+Br)Cr)^{1/2}}{2(Ar^3+Br)} \quad$ for $\quad t \in \bar{U}_r$.

Since $\quad y(0) = 0 \quad$ and $\quad \lim\limits_{r\to 0^+} \dfrac{1+(1-4(Ar^3+Br)Cr)^{1/2}}{2(Ar^3+Br)} = \infty$

there exists an r, $0 < r \leq T$ such that

$$|y'(t)| \leq \frac{1-(1-4(Ar^3+Br)Cr)^{1/2}}{2(Ar^3+Br)} \quad \text{for} \quad t \in \bar{U}_r.$$

Also the function $\quad h(r) = \dfrac{1-(1-4(Ar^3+Br)Cr)^{1/2}}{2(Ar^3+Br)} \quad$ is an

increasing function on $\left[0, \left[\dfrac{-BC+(B^2C^2+AC)^{1/2}}{2AC}\right]^{1/2}\right]$ so in particular

$$M(r) = \max_{t\in\bar{U}_r} |y'(t)| \leq \frac{1-(1-4(Ar^3+Br)Cr)^{1/2}}{2(Ar^3+Br)}$$

$$< \left[\frac{-2BC^2+2C(B^2C^2+AC)^{1/2}}{A}\right]^{1/2}.$$

Suppose $\bar{U}_r$ is the largest closed disc where

$$M(r) \leq \frac{1-(1-4(Ar^3+Br)Cr)^{1/2}}{2(Ar^3+Br)}.$$

If $\quad r < T$, then for $\quad r < \eta \leq T$

$$M(\eta) \geq \frac{1+(1-4(A\eta^3+B\eta)C\eta)^{1/2}}{2(A\eta^3+B\eta)} \geq \frac{1}{2(A\eta^3+B\eta)}$$

$$> \left[\frac{-2BC^2+2C(B^2C^2+AC)^{1/2}}{A}\right]^{1/2}$$

and this contradicts the fact that $M(p)$ is a continuous function of p for $0 \leq p \leq T$. Thus

$$|y'(t)| \leq \frac{1-(1-4(AT^3+BT)CT)^{1/2}}{2(AT^3+BT)} \equiv M_o \quad \text{for} \quad t \in \bar{U}_T$$

and so

$$|y(t)| \leqslant TM_o = M_1 \quad \text{for} \quad t \in \bar{U}_T.$$

Hence (4.9) implies

$$|y''(t)| \leqslant AM_1^2 + BM_o^2 + C \equiv M_2$$

and so $|y(t)|$, $|y'(t)|$, $|y''(t)| \leqslant K = \max\langle M_o, M_1, M_2\rangle$ and the existence of a solution is established by Theorem 4.7.

<u>Remark</u>. We can obtain corresponding theorems for the inhomogeneous case $n = 2$. Furthermore we can use the same ideas of section 4.3 to discuss higher order problems. An example of this is the following theorem:

<u>Theorem 4.10</u>. Suppose

$$|f(t,u_1,\ldots,u_{n-1})| \leqslant A_o(t)+A_1(t)|u_1|+\ldots+A_{n-1}(t)|u_{n-1}|$$

where $A_o(t),\ldots,A_{n-1}(t) \geqslant 0$ are functions bounded on bounded t sets. Let $A_o,\ldots,A_{n-1}$ be upper bounds for $A_o(t),\ldots,A_{n-i}(t)$ respectively for $t \in \bar{U}_T$. Then the homogeneous initial value problem (4.6) has at least one solution in $A^n(\bar{U}_T)$ provided $0 < T < T_o$ where T_o is the smallest positive root of $A_1 T^{n-1} + A^2\ T^{n-2} + \ldots + A_{n-1} T - 1 = 0$.

4.4 Remarks for Boundary Value Problems in the Complex Domain

It is also possible to consider "Boundary Value Problems" in the complex domain. A typical example of such a problem is:

$$4.10)\qquad \begin{cases} y'' = f(t,y,y'), & t \in U_T \\ y'(0) = 0 \text{ and } y(c) = 0; & c \in \bar{U}_T. \end{cases}$$

Trivial adjustments in the previous sections proofs yield

Theorem 4.11. Suppose

$$|f(t,u,p)| \leqslant A(t)|u|^2 + B(t)|p|^2 + C(t)$$

where $A(t)$, $B(t)$, $C(t) \geqslant 0$ are functions bounded on bounded t-sets. Let A,B and C denote the upper bounds for $A(t)$, $B(t)$, $C(t)$ respectively for $t \in \bar{U}_T$. Then (4.10) has at least one solution in $A^2(\bar{U}_T)$ provided $0 < T < \left\{\frac{-BC+(B^2C^2+2AC)^{1/2}}{4AC}\right\}^{1/2}$.

Remark. As usual we can consider inhomogeneous boundary conditions and higher order "Boundary Value Problems". It should also be noted that if the condition $y'(0) = 0$ in (4.10) is replaced by $y'(d) = 0$, $d \in \mathbb{C}$ then Theorem 4.11 holds as stated with $U_T = \{z : |z-d| < T\}$.

V. Weak Solutions to Initial and Boundary Value Problems

5.1 Introduction

Many physical situations give rise to initial and boundary value problems of the form

$$5.1)\qquad \begin{cases} y^{(n)} = f(t,y,y',\dots,y^{(n-1)}), & t \in [0,T] \\ y \in B \end{cases}$$

where in fact $f:[0,T] \times R^n \to R$ is discontinuous. Here of course B denotes suitable initial or boundary conditions. In this chapter we study the case where f satisfies the Caratheodory Conditions i.e.

(a) For fixed $(P_o,P_1,\dots,P_{n-1}) \in R^n$, $f(\cdot,P_o,P_1,\dots,P_{n-1})$ is lebesgue measurable on $[0,T]$.

(b) For almost all $t \in [0,T]$, $f(t,\cdot,\dots,\cdot)$ is continuous on R^n.

For notational purposes let $L^p(0,T)$, $p \geq 1$ a real number, denote the space of lebesgue measurable functions g on $(0,T)$ with $\int_0^T |g(t)|^p dt < \infty$. $L^p(0,T)$ with norm $\|g\|_{L^p} = \left[\int_0^T |g(t)|^p dt\right]^{1/p}$ is a Banach Space. By a weak solution to (5.1) we mean a function $y \in B$ which together with its derivatives $y',\dots,y^{(n-1)}$ are absolutely continuous on $[0,T]$ with $y^{(n)} \in L^1(0,T)$ and $y^{(n)} = f(t,y,\dots,y^{(n-1)})$ almost everywhere on $[0,T]$.

We shall establish, under reasonable physical assumptions on f, that (5.1) has bounded weak solutions. As before we restrict ourselves to the cases $n = 1$ and 2. Our analysis is based on the Topological Transversality Theorem and known results concerning Sobolev spaces and Nemysky operators.

5.2 Preliminary Notation and Results

Here we introduce some standard notation and collect together some known facts on Sobolev spaces and Nemysky operators which we will use throughout the remainder of this chapter.

First let ${}_nL^p = L^p \times L^p \times \ldots \times L^p$, n factors. Also let $H^k(0,T)$, $k > 0$ an integer, denote the space of all functions u on the interval $[0,T]$ which are absolutely continuous on $[0,T]$ together with their derivatives up to the order $k - 1$ and whose derivative $u^{(k)}$ (which exists almost everywhere) is an element of $L^1(0,T)$. $H^k(0,T)$ with norm

$$\|u\|_{H^k} = \sum_{j=0}^{k} \|u^{(j)}\|_{L^1}$$

is a Banach Space. As usual we let

$$H_B{}^k(0,T) = \{u \in H^k(0,T),\ u \in B\}.$$

For proofs and further information on the following facts see Royden [28], Fučík [12], Fučík and Kufner [13] and Vainberg [35].

<u>Theorem</u> <u>5.1</u>. Let g be a monotone increasing absolutely continuous function on $[a,b]$ with $g(a) = c$, $g(b) = d$. If f is a nonnegative measurable function on $[c,d]$ then

$$\int_c^d f(y)dy = \int_a^b f(g(x))\ g'(x)dx.$$

Theorem 5.2. (Sobolev Imbedding Theorem)

$H^k(a,b)$ is compactly imbedded into $C^{k-1}[a,b]$ i.e. the imbedding operator $j: H^k(a,b) \to C^{k-1}[a,b]$ is continuous and completely continuous.

Let Ω be a bounded and measurable set in R and let $u(t) = (u_1(t), u_2(t), \ldots, u_n(t))$, $t \in \Omega$.

Theorem 5.3. Suppose the real valued function $f(t, u_1(t), \ldots, u_n(t))$ satisfies the Caratheodory conditions and for $p_1, p_2, \ldots, p_n \geq 1$ there exists a constant $B > 0$ and a function $g \in L^1(\Omega)$ such that for any $u \in R^n$ and almost all $t \in \Omega$ we have

$$|f(t,u_1,\ldots,u_n)| \leq g(t) + B \sum_{k=1}^{n} |u_k|^{p_k}.$$

Then the Nemysky operator F defined by $F[u](t) = f(t,u_1(t),\ldots,u_n(t))$ is a continuous operator from $L^{p_1} \times \ldots \times L^{p_n}$ into L^1.

Finally let Ω and u be as above and now let the real valued function $f_i(t,u_1(t),\ldots,u_n(t))$, $i=1,\ldots,n$ satisfy the Caratheodory Conditions. Now each function f_i generates an operator F_i defined by $F_i[u](t) = f_i(t,u_1(t),\ldots,u_n(t))$, $t \in \Omega$. Let $F = (F_1,\ldots,F_n)$.

Corollary 5.4. Suppose for $p_1,\ldots,p_n \geq 1$ there exists a constant $B > 0$ and a function $g_i \in L^1(\Omega)$ such that for any $u \in R^n$ and almost all $t \in \Omega$ we have

$$|f_i(t,u_1,\ldots,u_n)| \leqslant g_i(t) + B \sum_{k=1}^{n} |u_k|^{p_k}.$$

Then the mapping F is a continuous operator from $L^{p_1} \times L^{p_2} \times \ldots \times L^{p_n}$ into ${}_nL^1$.

5.3 Weak Solutions to First Order Initial Value Problems

The Sobolev Imbedding Theorem and the continuity of the Nemysky operator are used to extend Theorem 1.9 for the new class of problems (5.1).

Theorem 5.5. Let $f:[0,T] \times R \to R$ satisfy the Caratheodory Conditions and $0 \leqslant \lambda \leqslant 1$. Suppose also for $p \geqslant 1$,

5.2) $$|f(t,y)| \leqslant A(t)\,|y|^p + B(t)$$

for bounded functions $A(t)$, $B(t) \geqslant 0$. Let A_o and B_o be upper bounds for $A(t)$, $B(t)$ respectively. Finally we assume that there is a constant K such that $\|y\|_{H^1} \leqslant K$ for each solution $y(t)$ to

5.3)$_\lambda$ $$\begin{cases} y' = \lambda f(t,y), & t \in [0,T] \\ y(0) = 0. \end{cases}$$

Then the initial value problem

5.3) $$\begin{cases} y' = f(t,y), & t \in [0,T] \\ y(0) = 0 \end{cases}$$

has a solution y in $H^1(0,T)$.

Proof As before B denotes the set of functions which satisfy the initial condition $y(0) = 0$. Let

$\bar{V} = \{u \in H_B^1(0,T);\ \|u\|_{H^1} \leqslant K+1\}$ and define

$F_\lambda : L^p(0,T) \to L^1(0,T)$, $0 \leqslant \lambda \leqslant 1$, by $F_\lambda[v](t) = \lambda f(t,v(t))$. Now F_λ is a continuous operator from $L^p(0,T)$ into $L^1(0,T)$ by Theorem 5.3. We also have the imbedding $j : H_B^1(1,T) \to C[0,T]$ defined by $ju = u$ completely continuous by Theorem 5.2. Now let $T : C[0,T] \to L^p(0,T)$ be the continuous operator defined by $Tu = u$. Finally we define $N : H_B^1(0,T) \to L^1(0,T)$ by $Ny = y'$. Clearly N is linear and continuous. N is also one-to-one since if $Ny = 0$ the absolute continuity of y together with the initial condition yields $y \equiv 0$. To show N is onto let

$f(t) \in L^1(0,T)$ and take $y(t) = \int_0^t f(u)du$. Clearly y

is absolutely continuous, $y(0) = 0$, $y' = f(t)$ almost everywhere and $y' \in L^1(0,T)$. It follows from Theorem 1.6 that N^{-1} is a bounded linear operator. These operators and their relationships are conveniently displayed:

$$\begin{tikzcd}
C[0,T] \arrow[r, "T"] & L^p(0,T) \arrow[d, "F_\lambda"] \\
H_B^1(0,T) \arrow[u, "j"] & L^1(0,T) \arrow[l, "N^{-1}"]
\end{tikzcd}$$

Figure 1

Thus $H_\lambda = N^{-1}F_\lambda Tj : \bar{V} \to H_B^1(0,T)$ defines a homotopy. It is clear that the fixed points of H_λ are precisely the solutions to $(5.3)_\lambda$. Now H_λ is fixed point free on ∂V. Moreover, the complete continuity of j together with the continuity of N^{-1}, F_λ and T imply that the homotopy H_λ is compact. Now H_o is essential so Theorem 1.5 implies that H_1 is essential. Thus (5.3) has a solution.

We are now in a position to prove our basic existence theorem for first order initial value problems.

<u>Theorem 5.6</u>. Suppose $f:[0,T] \times R \to R$ satisfies the Caratheodory Conditions and (5.2). Then the initial value problem (5.3) has a solution in $H^1(0,T)$ for

$$T < \int_0^\infty \frac{du}{A_o \; u^P + B_o} \, .$$

<u>Proof</u> To prove existence of a solution in $H^1(0,T)$ we apply Theorem 5.5. So to establish the a priori bounds for $(5.3)_\lambda$, let $y(t)$ be a solution to $(5.3)_\lambda$. Then since

$$|\lambda f(t,y)| \leqslant A_o \; |y|^P + B_o$$

we have for each $t \in [0,T]$

$$\int_0^t \frac{|y'(u)|}{A_o |y(u)|^P + B_o} \, du \leqslant t \leqslant T.$$

Now $|y(u)| = \left|\int_0^u y'(s)ds\right| \leqslant \int_0^u |y'(s)|ds$

So $$A_o \; |y(u)|^P + B_o \leqslant A_o \left(\int_o^u |y'(s)|ds\right)^P + B_o$$

since $A_o \; x^P + B_o$ is an increasing function for $x \geqslant 0$.

Thus the previous inequality implies

$$\int_0^t \frac{|y'(u)|}{A_o\left[\int_0^u |y'(s)|ds\right]^p + B_o}\, du \leqslant T < \int_0^\infty \frac{du}{A_o\ u^p + B_o} .$$

Theorem 5.1 with $g(u) = \int_0^u |y'(s)|ds$ yields

$$\int_0^{g(t)} \frac{dx}{A_o\ x^p + B_o} \leqslant T < \int_0^\infty \frac{du}{A_o\ u^p + B_o} .$$

So there exists a constant $M < \infty$ such that $g(t) \leqslant M$. Moreover,

$$|y(t)| \leqslant g(t) \leqslant M \quad \text{so} \quad \int_0^T |y(t)|dt \leqslant MT.$$

Also (5.2) and the differential equation yield

$$\int_0^T |y'(t)|dt \leqslant A_o\left[\int_0^T |y(t)|^p dt\right] + B_o\ T \equiv M_1$$

for some constant $M_1 < \infty$. So $\|y\|_{H^1} \leqslant K = MT + M_1$, and the existence of a solution to (5.3) is established.

Theorem 5.5 also holds for the inhomogeneous initial condition $y(0) = r$, via ideas of Theorem 1.10. So trivial adjustments in the proof of Theorem 5.6 yields:

<u>Theorem</u> <u>5.7</u>. Suppose $f:[0,T] \times R \to R$ satisfies the

Caratheodory Conditions and (5.2). Then the initial value problem

$$\begin{cases} y' = f(t,y), \ t \in [0,T] \\ y(0) = r \end{cases}$$

has a solution in $H^1(0,T)$ for each $T < \int_{|r|}^{\infty} \frac{du}{A_o \ u^p + B_o}$.

Example 5.1. (Electrical Circuits)

Lets consider an electrical circuit which contains a resistance R, a condenser of capacitance C, a switch S and a generator E. Suppose the switch is closed at $t = 0$. Then Kirchoff's, Ohm's and Coulomb's Laws imply that the RC circuit satisfies

$$R \frac{dq}{dt} + \frac{1}{C} q(t) = E(t,q)$$

where $q(t)$ denotes the charge on the capacitor at time t and $E(t,q)$ the value of the voltage impressed on the circuit by E.

We also have $q(0) = q_o$ where q_o is the charge on the capacitor at $t = 0$. Thus we are interested in the initial value problem

5.4)
$$\begin{cases} \frac{dq}{dt} = - \frac{1}{RC} q + \frac{E(t,q)}{R} \\ q(0) = q_o \end{cases}$$

Suppose $|E(t,q)| \leqslant A(t)|q| + B(t)$ for bounded functions $A(t)$, $B(t) \geqslant 0$ and that $E(t,q)$ satisfies the Caratheodory conditions. Consequently since

$$\int_{|q_o|}^{\infty} \frac{dq}{A_o q + B_o} = \infty, \quad \text{for any constants } A_o, B_o > 0,$$

Theorem 5.7 implies that (5.4) has a solution in

$H^1(0,T)$ for any $T > 0$. On the other hand suppose

$$|E(t,q)| \leqslant A(t)\ |q|^m + B(t),\ m = 1,2,\ldots\ .$$

for bounded functions $A(t)$, $B(t) \geqslant 0$ and that $E(t,q)$ satisfies the Caratheodory conditions. Then Theorem 5.7 implies that (5.4) has a solution in $H^1(0,T)$ for any $T < T_\infty$ where T_∞ is as described in Example 1.2.

5.4 Weak Solutions to Boundary Value Problems

In this section we examine problems of the form

$$\text{5.5)} \qquad \begin{cases} y'' = f(t,y,y'), & t \in [0,1] \\ y \in B \end{cases}$$

where f is defined on $[0,1] \times R \times R$. Here B denotes either the boundary conditions

(i) $y(0) = 0,\ y(1) = 0$

or

(ii) $-\alpha y(0) + \beta y'(0) = 0; \alpha, \beta > 0$
$\qquad ay(1) + by'(1) = 0; a,b > 0.$

We again use the results of Section 5.2 and the Topological Transversality Theorem to extend Theorem 1.9 for the class of problems described above.

<u>Theorem</u> <u>5.8</u>. Let $f:[0,1] \times R \times R \rightarrow R$ satisfy the Caratheodory Conditions and $0 \leqslant \lambda \leqslant 1$. Suppose also for $m \geqslant 1$

$$|f(t,y,p)| \leqslant A(t)\left[|y|^m + p^2\right] + B(t)$$

where $A(t)$, $B(t) \geqslant 0$ are bounded functions. Let A and B be upper bounds for $A(t)$, $B(t)$ respectively. Finally we assume that there is a constant K such that

$\|y\|_{H^2} \leqslant K$ for each solution $y(t)$ to

$$5.5)_\lambda \qquad \begin{cases} y'' = \lambda f(t,y,y'), & t \in [0,1] \\ y \in B. \end{cases}$$

Then the boundary value problem (5.5) has a solution y in $H^2(0,1)$.

<u>Proof</u> Let $\bar{V} = \{u \in H_B^2(0,1) : \|u\|_{H^2} \leqslant K + 1\}$ and define $F_\lambda : L^m(0,1) \times L^2(0,1) \to L^1(0,1)$, $0 \leqslant \lambda \leqslant 1$, by $F_\lambda(v,v')(t) = \lambda f(t,v(t),v'(t))$. Now F_λ is a continuous operator from $L^m(0,1) \times L^2(0,1)$ into $L^1(0,1)$ by Theorem 5.3. Again the imbedding $j : H_B^2(0,1) \to C^1[0,1]$ defined by $ju = u$ is completely continuous by Theorem 5.2. We also let $T : C^1[0,1] \to L^m(0,1) \times L^2(0,1)$ be the continuous operator defined by $Tu = (u,u')$ Finally we define $N : H_B^2(0,1) \to L^1(0,1)$ by $Ny = y''$. It is easy to check that N is linear, onto and continuous. To show N is one-to-one we observe that the boundary conditions (i) or (ii) imply that y' vanishes at least once in $[0,1]$. So if $Ny = 0$ the absolute continuity of y and y' with the above observation implies $y \equiv 0$. Thus N^{-1} is a bounded linear operator by Theorem 1.6. These operators and their relationships are conveniently displayed:

$$\begin{tikzcd} C^1[0,T] \arrow[r, "T"] & L^m(0,1) \times L^2(0,1) \arrow[d, "F_\lambda"] \\ H_B^2(0,1) \arrow[u, "j"] & L^1(0,1) \arrow[l, "N^{-1}"'] \end{tikzcd}$$

Figure 2

Thus $H_\lambda = N^{-1}F_\lambda Tj$ defines a homotopy $H_\lambda:\bar{V} \to H_B^2(0,1)$. It is clear that the fixed points of H_λ are precisely the solutions to $(5.5)_\lambda$. Now H_λ is fixed point free on ∂V. Moreover, the complete continuity of j together with the continuity of N^{-1}, F_λ and T imply that the homotopy H_λ is compact. Now H_o is essential so Theorem 1.5 implies that H_1 is essential. Thus (5.5) has a solution.

Next sufficient conditions on f are given which imply a priori bounds for solutions to (5.5). Let $y(t) \in H_B^2(0,1)$ be a solution to (5.5). Suppose $[y(t)]^2$ has a maximum at $t_o \in (0,1)$. Then from elementary calculus $y'(t_o) = 0$.

<u>Theorem</u> <u>5.9</u>. Suppose f satisfies the following:

5.7) $\begin{cases} \text{There is a constant } M \geqslant 0 \text{ such that} \\ yf(t,y,0) > 0 \text{ for } |y| > M. \end{cases}$

5.8) $\begin{cases} yf(t,y,p) \text{ is lower semicontinuous on} \\ [0,i] \times R \times R. \end{cases}$

Then any solution y to (5.5) satisfies

$$|y(t)| \leqslant M, \ t \in [0,1].$$

<u>Proof</u> We first show that $|y|$ cannot have a nonzero maximum at 0 or 1. This is true automatically if y satisfies (i). On the other hand suppose y satisfies (ii) and that $|y|$ has a nonzero maximum at 0. Then $y(0)\ y'(0) \leqslant 0$. However, $y(0)\ y'(0) = \frac{\beta}{\alpha}\ [y(0)]^2 > 0$, a contradiction. A similar argument works for the case $t = 1$. We conclude that $|y|$ can only have a nonzero maximum at $t_o \in (0,1)$. Now assume the maximum of $|y|$ is at $t_o \in (0,1)$, so $y'(t_o) = 0$. Suppose $|y(t_o)| > M$. Then from (5.7),

$y(t_o)\ f(t_o,\ y(t_o),\ 0) > 0$. The continuity of y and y' together with (5.8) implies there exists a neighborhood N_{t_o} of $(t_o, y(t_o), 0)$ such that

$$(5.9)\quad y(t)f(t,y(t),y'(t)) > 0 \quad \text{for} \quad (t,y(t),y'(t)) \in N_{t_o}.$$

On the other hand $y'(t) = \int_{t_o}^{t} y''(s)ds$ and so Fubini's Theorem implies

$$y(t) = y(t_o) + \int_{t_o}^{t} (t - u)\ y''(u)du.$$

Thus

$$y^2(t) = y^2(t_o) + 2\int_{t_o}^{t} (t-u)\Big[y(u)f(u,y(u),y'(u)) + [y'(u)]^2\Big]du.$$

Since $|y|$ has a maximum at t_o then for t near t_o

$$\int_{t_o}^{t} (t-u)\Big[y(u)f(u,y(u),y'(u)) + [y'(u)]^2\Big]du \leq 0$$

which contradicts (5.9). Thus $|y(t_o)| \leq M$.

We now prove our basic existence theorem for second order boundary value problems.

<u>Theorem 5.10</u>. Suppose $f:[0,1] \times R \times R \to R$ satisfies the Caratheodory Conditions, (5.6), (5.7) and (5.8). Then the boundary value problem (5.5) has at least one solution in $H^2(0,1)$.

<u>Proof</u> To prove existence of a solution in $H^2(0,1)$ we apply Theorem 5.8. To establish a priori bounds for $(5.5)_\lambda$, let $y(t)$ be a solution to $(5.5)_\lambda$. If $\lambda = 0$

we have the unique solution $y \equiv 0$. Otherwise for $0 < \lambda \leq 1$, $yf(t,y,0) > 0$ for $|y| > M$ implies $\lambda yf(t,y,0) > 0$ for $|y| > M$. Thus Theorem 5.9 implies $|y| \leq M$ for any solution y to $(5.5)_\lambda$ and for each $\lambda \in [0,1]$. Hence $\int_0^1 |y(s)|ds \leq M$. Finally we obtain a priori bounds on derivatives of y. We have already observed that boundary conditions (i) or (ii) imply that y' vanishes at least once on $[0,1]$, so each point $t \in [0,1]$ for which $y'(t) \neq 0$ belongs to an interval $[\mu,v]$ such that y' maintains a fixed sign on $[\mu,v]$ and $y'(\mu)$ and/or $y'(v)$ is zero. Assume that $y'(\mu) = 0$ and $y' \geq 0$ on $[\mu,v]$. Thus, with $B_o = AM^m + B$, $A_o = A$ where A and B denote the upper bounds for $A(t)$, $B(t)$ respectively and since

$$|\lambda f(t,y,y')| \leq A_o(y')^2 + B_o,$$

we have

$$\int_\mu^t \frac{y'(u)|y''(u)|}{A_o[y'(u)]^2+B_o}\, du \leq 2M.$$

For $\mu \leq u \leq t$

$$[y'(u)]^2 = |[y'(u)]^2| = 2\left|\int_\mu^u y'(s)y''(s)ds\right| \tag{5.10}$$

$$\leq 2\int_\mu^u y'(s)|y''(s)|ds,$$

so $$A_o[y'(u)]^2 + B_o \leq 2A_o\int_\mu^u y'(s)|y''(s)|ds + B_o.$$

Thus the previous inequality implies

$$\int_{\mu}^{t} \left[\frac{2\, A_o\, y'(u)|y''(u)|}{2A_o \int_{\mu}^{u} y'(s)|y''(s)|ds + B_o} \right] du \leqslant 4A_oM.$$

Theorem 5.1 with $g(u) = 2A_o \int_{\mu}^{u} y'(s)|y''(s)|ds$ yields

$$\int_{0}^{g(t)} \frac{du}{u+B_o} \leqslant 4A_oM, \quad \text{and so}$$

$g(t) \leqslant B_o(e^{4A_oM} - 1)$. Moreover (5.10) yields

$$[y'(t)]^2 \leqslant 2\int_{\mu}^{t} y'(s)\ |y''(s)|ds \leqslant \frac{B_o}{A_o}(e^{4A_oM} - 1)$$

and so

$$|y'(t)| \leqslant \left[\frac{B_o}{A_o}\ (e^{4A_oM} - 1)\right]^{1/2} \equiv M_1.$$

The other cases are treated similarly and the same bound is obtained. Thus $|y'| \leqslant M_1$ and in particular $\int_0^1 |y'(s)|ds \leqslant M_1$ for each solution y to $(5.5)_\lambda$ and for each $\lambda \in [0,1]$. Also (5.6) and the differential equation yields

$$\int_0^1 |y''(t)|dt \leqslant A_o\int_0^1 |y'(t)|^2 dt + B_o \leqslant A_oM_1^2 + B_o \equiv M_2.$$

So $\|y\|_{H^2} \leqslant K = M_o + M_1 + M_2$ and the existence of a solution to (5.5) is established.

Our arguments in Theorem 5.9 only uses (5.7), (5.8) to conclude that $y(t)f(t,y(t),y'(t))+[y'(t)]^2>0$ in a neighborhood of t_o where $|y(t_o)|$, $t_o \in (0,1)$, is the maximum of $|y|$; so in fact we have also shown the following theorem:

<u>Theorem</u> <u>5.11</u>. Suppose $f:[0,1] \times R \times R \to R$ satisfies the Caratheodory Conditions and (5.6). Suppose in addition that there is a constant $M \geq 0$ such that

$$yf(t,y,p) + p^2 > 0 \quad \text{for} \quad |y| > M.$$

then the boundary value problem (5.5) has at least one solution in $H^2(0,1)$.

We also have analogue results for the inhomogeneous problem

$$\text{5.11)} \qquad \begin{cases} y'' = f(t,y,y'), & t \in [0,1] \\ y \in B \end{cases}$$

where B denotes either the boundary conditions

(iii) $y(0) = r$, $y(1) = s$

or

(iV) $-\alpha y(0) + \beta y'(0) = r;\ \alpha,\beta > 0$
$\quad\ ay(1) + by'(1) = s;\ a,b > 0.$

Extending the ideas of Theorem's 1.10, 5.8 and 5.10 we obtain:

<u>Theorem</u> <u>5.12</u>. Suppose f satisfies the Caratheodory Conditions and (5.6), (5.7), (5.8). Then the boundary value problem (5.11) has at least one solution in $H^2(0,1)$.

<u>Example</u> <u>5.2</u> (Heat Conduction)

Suppose V is an isotropic heat conducting medium with S denoting the surface and $\vec{n}$ the outer normal. We define $u = u(x,t)$ to be the temperture at location $x \in V$ and time $t > 0$. Also $c = c(x,u)$ denotes the specific heat, $p = p(x,u)$ the density and $k = k(x,u)$ the thermal conductivity. Now the Divergence Theorem and Fourier's Law together with conservation of energy yields the heat equation

$$\frac{\partial}{\partial t}(cpu) = \operatorname{div}(k\vec{\nabla}u) + h; \; x \in v, \; t > 0$$

where $h = h(x,u)$ represents the rate of heat generation by internal sources.

We now set up boundary conditions which describe the heat transfer across S. Suppose the surroundings of V are kept at a time independent temperature and that heat radiates into the surroundings (according to Newton's Law of Cooling) at a rate proportional to the temperature difference between S and its surrounding environment. The energy balance of heat flow across S together with Fourier's Law yields

$$z(x,u)u(x,t) + \sigma(x,u)\,\frac{\partial u(x,t)}{\partial n} = g(x); \; x \in S, \; t > 0$$

where $z \geqslant 0$, $\sigma \geqslant 0$ and $\sigma + z > 0$.
We wish to find a steady state solution (temperature distribution) $y = y(x)$. It will satisfy

$$\begin{cases} \Delta y = -\frac{1}{k}\left[\vec{\nabla}k . \vec{\nabla}y + h\right], & x \in v \\ z(x,y)y(x) + \sigma(x,y)\,\frac{\partial u(x)}{\partial n} = g(x), & x \in S. \end{cases}$$

Now if V is a rod of unit length and insulated laternal surfaces then the steady state problem is

$$\begin{cases} y'' = -\frac{1}{k(x,y)}\,[k_x(x,y)y' + k_y(x,y)(y')^2 + h(x,y)] \\ z(0,y(0))y(0) - \sigma(0,y(0))y'(0) = g(0) \\ z(1,y(1))y(1) + \sigma(1,y(1))y'(1) = g(1). \end{cases}$$

We will assume the case where z,σ are independent of temperature and set $\alpha = z(0) > 0$, $\beta = \sigma(0) > 0$, $a = z(1) > 0$, $b = \sigma(1) > 0$, $r = g(0)$ and $s = g(1)$. So our problem reduces to

$$\text{5.12)}\qquad \begin{cases} y'' = -\frac{1}{k(x,y)}\left[k_x\; y' + k_y(y')^2 + h(x,y)\right] \equiv f(x,y,y') \\ \alpha y(0) - \beta y'(0) = r;\ \alpha,\beta > 0 \\ ay(1) + by'(1) = s;\ a,b > 0. \end{cases}$$

Now we make the following assumptions on h and k:

5.13) $\begin{cases} k_x(x,y),\ k_y(x,y) \text{ are continuous and} \\ \text{bounded for } (x,y) \in [0,1] \times R. \text{ Also suppose} \\ \text{for } (x,y) \in [0,1] \times R,\ k(x,y) \\ \text{is continuous and } k(x,y) \geqslant m > 0 \\ \text{where } m \text{ is a constant.} \end{cases}$

5.14) $\begin{cases} \text{Suppose there are constants } A,B > 0 \text{ such} \\ \text{that } |h(x,y)| \leqslant A|y|^{\alpha} + B,\ x \in [0,1], \\ \text{for some constant } \alpha \geqslant 1. \text{ Suppose also } h \\ \text{satisfies the Caratheodory Conditions.} \end{cases}$

5.15) $\qquad yh(x,y) < 0$ for large $|y|$.

5.16) $\qquad \frac{y(x)}{k(x,y)}\,h(x,y)$ is lower semicontinuous.

The assumption (5.15) that $yh(x,y) < 0$ for large $|y|$ means that the internal heat generation $h(x,y)$ opposes large temperature extremes i.e. if $y > 0$ and $|y|$ large, then $h(x,y) < 0$ so heat is removed from the rod by internal sources and the temperature tends to drop.

Now assumptions (5.13), (5.14), (5.15), (5.16) together with Theorem 5.12 implies that (5.12) has at least one solution in $H^2(0,1)$.

5.5 Weak Solutions to Systems of First Order Initial Value Problems

We now establish existence theorems for the class of problems described by

$$5.17)\qquad \begin{cases} y' = f(t;y), & t \in [0,T] \\ y(0) = 0 \end{cases}$$

where $f:[0,T] \times R^n \to R^n$ is defined and satisfies the Caratheodory Conditions i.e.

(a) For fixed $y = (y_1,\ldots,y_n) \in R^n$, $f(\cdot;y)$ is measurable on $[0,T]$.

(b) For almost all $t \in [0,T]$, $f(t;\cdot)$ is continuous on R^n.

<u>Theorem</u> <u>5.13</u>. Let $f:[0,T] \times R^n \to R^n$ satisfy the Caratheodory Conditions and $0 \leq \lambda \leq 1$. Suppose also for $p \geq 1$

$$5.18)\qquad |f(t;y)| \leq A(t)\ |y|^p + B(t)$$

for bounded functions $A(t)$, $B(t) \geq 0$. Let A_o and B_o be upper bounds for $A(t), B(t)$ respectively. Finally we assume that there is a constant K such that $\|y\|_{H^1} \leq K$ for each solution $y(t)$ to

$$5.17)_\lambda\qquad \begin{cases} y' = \lambda f(t;y), & t \in [0,T] \\ y(0) = 0. \end{cases}$$

Then the initial value problem (5.17) has a solution in $H^1(0,T)$.

<u>Proof</u> As before B denotes the set of functions which satisfy the initial condition $y(0) = 0$. Let $\bar{V} = \{u \in H^1_B(0,T) : \|u\|_{H^1} \leqslant K+1\}$ and define

$F_\lambda : {}_nL^p(0,T) \to {}_nL^1(0,T)$, $0 \leqslant \lambda \leqslant 1$, by $F_\lambda[v](t) = \lambda f(t;v(t))$. Now F_λ is a continuous operator from ${}_nL^p(0,T)$ into ${}_nL^i(0,T)$ by Corollary 5.4. Let $T : C[0,T] \to {}_nL^p(0,T)$ be defined by $Tu = (u_1,\ldots.,u_n)$ where $u = (u_1,\ldots.,u_n)$. T is continuous since

$$\|Tu\|_{{}_nL^p} = \|u_1\|_{L^p} + \ldots + \|u_n\|_{L^p} \leqslant n\|u\|_o \, T^{1/p}.$$

We also have the imbedding $j : H^1_B(0,T) \to C[0,T]$ defined by $ju = u$ completely continuous by Theorem 5.2. Finally we define $N : H^1_B(0,T) \to {}_nL^1(0,T)$ by $Ny = y'$. It follows from Theorem 1.6 that N^{-1} is a bounded linear operator.

$$\begin{tikzcd}
C[0,T] \arrow[r, "T"] & {}_nL^p(0,T) \arrow[d, "F_\lambda"] \\
H^1_B(0,T) \arrow[u, "j"] & {}_nL^1(0,T) \arrow[l, "N^{-1}"]
\end{tikzcd}$$

Figure 3

Thus $H_\lambda = N^{-1}F_\lambda Tj : \bar{V} \to H^1_B(0,T)$ defines a homotopy. It is clear that the fixed points of H_λ are precisely the solutions to $(5.17)_\lambda$. Thus H_λ is fixed point free on

∂V. Finally since H_λ is compact and H_o is essential we have from Theorem 1.5 that H_1 is essential. Thus (5.17) has a solution.

We have the exact analogue of Theorem 5.6 for systems of first order initial value problems.

Theorem 5.14. Let $f:[0,T] \times R^n \to R^n$ satisfy the Caratheodory Conditions and (5.18). Then (5.17) has a solution in $H^1(0,T)$ for $T < \int_0^\infty \frac{du}{A_o u^P + B_o}$.

Proof To prove existence of a solution in $H^1(0,T)$ we apply Theorem 5.13. To establish the a priori bounds for $(5.17)_\lambda$, let $y(t)$ be a solution to $(5.17)_\lambda$. Now since $y(0) = 0$, each point t for which $y(t) \neq 0$ belongs to an interval $[a,t]$ in $[0,T]$ such that $|y(s)| > 0$ on $a < s \leqslant t$ and $y(a) = 0$. Thus for any point t where $y(t) \neq 0$

$$\int_a^t \Big| |y(s)|' \Big| ds = \int_a^t \frac{|y(s) \cdot y'(s)|}{|y(s)|} ds \leqslant \int_a^t |y'(s)| ds$$

and since $|\lambda f(t;y)| \leqslant A_o |y|^P + B_o$ we have

$$\int_a^t \frac{\big||y(s)|'\big|}{A_o |y(s)|^P + B_o} ds \leqslant t-a \leqslant T < \int_0^\infty \frac{du}{A_o u^P + B_o}.$$

Now $$|y(s)| = \left|\int_a^s |y(u)|' du\right| \leqslant \int_a^s \Big| |y(u)|' \Big| du,$$

so $$A_o |y(s)|^P + B_o \leqslant A_o \left[\int_a^s \Big| |y(u)|' \Big| du\right]^P + B_o.$$

Consequently the previous inequality implies

$$\int_a^t \left[\frac{||y(s)|'|}{A_o\left(\int_a^s \left| |y(u)|' \right| du\right)^p + B_o}\right] ds \leq T < \int_0^\infty \frac{du}{A_o u^p + B_o}.$$

Theorem 5.1 with $g(s) = \int_a^s \left| |y(u)|' \right| du$ yields

$$\int_0^{g(t)} \frac{dx}{A_o x^p + B_o} \leq T < \int_0^\infty \frac{du}{A_o u^p + B_o}.$$

So there exists a constant $M < \infty$ such that $g(t) \leq M$. Moreover,

$|y(t)| = \left| \int_a^t |y(u)|' du \right| \leq g(t) \leq M$ and in particular

$\int_0^T |y(t)| dt \leq MT$. Also (5.18) and the differential equation yields

$$\int_0^T |y'(t)| dt \leq A_o\left(\int_0^T |y(t)|^p dt\right) + B_o T \equiv M_1$$

for some constant $M_1 < \infty$. So $\|y\|_{H^1} \leq K = MT + M_1$ and the existence of a solution to (5.17) is established.

Analogue results hold for the inhomogeneous initial condition $y(0) = r$. Extending the ideas of Theorem's 1.10, 5.13 and 5.14 we obtain:

<u>Theorem</u> <u>5.15</u>. Suppose f satisfies the Caratheodory Conditions and (5.18). Then

$$\begin{cases} y' = f(t;y), & t \in [0,T] \\ y(0) = r \end{cases}$$

has a solution in $H^1(0,T)$ provided

$$T < \int_{|r|}^{\infty} \frac{du}{A_o u^p + B_o}.$$

VI. Third Order Boundary Value Problems

6.1 Introduction

We are interested in this chapter in studying the existence of solutions to third order boundary value problems of the form

$$6.1) \qquad \begin{cases} y''' = f(t,y,y'y''), & t \in [0,1] \\ y \in B \end{cases}$$

where $f:[0,1] \times R^3 \to R$ is continuous. Here B denotes suitable boundary conditions.

In section 6.2 results of Granas, Guenther and Lee [15], [16] on second order boundary value problems are extended so that existence theorems can be obtained for a certain class of third order boundary value problems. The existence theorems obtained in section 6.2 however are rather specialized. In section 6.3 by placing different types of monotonicty and growth conditions on the nonlinearity f, we obtain new and interesting existence theorems for a wide class of problems.

6.2 The Bernstein Theory of the Equation $y''' = f(t,y,y',y'')$

In this section we extend the Bernstein Theory and results of Granas, Guenther and Lee [15] to discuss problems of the form (6.1). Fix a point c in [0,1]. Let B denote either the boundary conditions

(i) $\quad y(c) = 0, \; y'(0) = 0, \; y'(1) = 0$

(ii) $\quad y(c) = 0, \; y''(0) = 0, \; y''(1) = 0$

or

(iii) $\quad sy(c) + dy'(c) = 0, \; s \neq 0$

$$-\alpha y'(0) + \beta y''(0) = 0, \; \alpha, \beta > 0$$

$$ay'(1) + by''(1) = 0, \; a, b > 0.$$

Theorem 1.9 was proven for two point boundary value problems; however, no change in the proof is necessary if we consider multipoint boundary value problems. Hence specializing Theorem 1.9 for the case $n = 3$ we obtain:

Theorem 6.1. Let $f:[0,1] \times R^3 \to R$ be continuous and $0 \leq \lambda \leq 1$. Suppose there is a constant K such that $\|y\|_3 \leq K$ for each solution $y(t)$ to

$$(6.1)_\lambda \qquad \begin{cases} y''' - y' = \lambda[f(t,y,y',y'') - y'], \; t \in [0,1] \\ y \in B. \end{cases}$$

Then the boundary value problem (6.1) has at least one solution in $C^3[0,1]$.

Next sufficient conditions on f are given which imply an a priori bound on any solution y and its derivative y' to (6.1). Suppose $y(t)$ is a solution to (6.1) and $[y'(t)]^2$ has a maximum at $t_o \in (0,1)$. Then

$$(6.2) \quad y''(t_o) = 0 \text{ and } y'(t_o)\, f(t_o, y(t_o), y'(t_o), 0) \leq 0.$$

Theorem 6.2. Suppose there is a constant $M \geq 0$ such that

$$pf(t,u,p,0) > 0 \quad \text{for} \quad |p| > M,$$

and (t,u) in $[0,1] \times R$.

Then any solution y to (6.1) satisfies

$$|y'(t)| \leq M \quad \text{for} \quad t \in [0,1].$$

Furthermore, there exists a constant $M_1 < \infty$ such that

$$|y(t)| \leq M_1 \quad \text{for} \quad t \in [0,1].$$

In fact $M_1 = M$ if y satisfies (i) or (ii) and $M_1 = M\left(1 + \left|\frac{d}{s}\right|\right)$ if y satisfies (iii).

<u>Proof</u> Suppose first $|y'|$ achieves a maximum at $t_o \in (0,1)$. Assume $|y'(t_o)| > M$. Then $y'(t_o)f(t_o,y(t_o),y'(t_o),0) > 0$, which contradicts (6.2). Thus $|y'(t_o)| \leq M_o$.

We next show that $|y'|$ cannot have a nontrivial maximum at 0 or 1 if $y(t)$ satisfies (i) or (iii). This is trivially true if y satisfies (i). Now suppose y satisfies (iii) and the maximum of $|y'|$ occurs at 0. Then $y'(0)\ y''(0) \leq 0$. However, from (iii) $y'(0)\ y''(0) = \frac{\beta}{\alpha}\ [y'(0)]^2 > 0$, a contradiction. A similar argument works for the case $t = 1$.

Finally we show that if y is solution to (ii) and if $|y'|$ assumes its maximum at $t_o = 0$ or $t_o = 1$ then $|y'(t_o)| \leq M$. Suppose y satisfies (ii) and $|y'(0)|$ is the maximum value of $|y'|$. If we assume $|y'(0)| > M$, then $y'(0)\ y'''(0) > 0$. Now if $y'(0) > 0$ then $y'''(0) > 0$, so $y''(t) = \int_0^t y'''(z)dz$ is strictly increasing near $t = 0$. We then have $y''(t) > y''(0) = 0$ for $t > 0$ and near zero and so $|y'(0)| = y'(0)$ is not the maximum of $|y'|$ on $[0,1]$, a contradiction. We obtain a similar contradiction if we assume $y'(0) < 0$. Thus $|y'(0)| \leq M$. Now if y satisfies (ii) and $|y'(1)|$ is the maximum value of $|y'|$, then we can show in the same way that $|y'(1)| \leq M$. Hence $|y'(t)| \leq M$ for $t \in [0,1]$. Also if y satisfies (i) or (ii), $|y(t)| = \left|\int_c^t y'(z)dz\right| \leq M$ while if y satisfies (iii), $|y(t)| \leq \left|\int_c^t y'(z)dz\right| + |y(c)|$ $\leq M\left(1 + \left|\frac{d}{s}\right|\right)$ for $t \in [0,1]$. This proves Theorem 6.2.

We couple the monotonicity condition in Theorem 6.2 with the analogue of the Bernstein growth condition to obtain our basic existence theorem.

<u>Theorem</u> <u>6.3</u>. Let $f:[0,1] \times R^3 \to R$ be continuous.

(a) Suppose there is a constant $M \geqslant 0$ such that
$$pf(t,u,p,0) > 0 \quad \text{for} \quad |p| > M,$$
and $(t,u) \in [0,1] \times R$.

(b) Suppose that
$$|f(t,u,p,q)| \leqslant A(t,u,p)\, q^2 + B(t,u,p)$$
where $A(t,u,p)$, $B(t,u,p) \geqslant 0$ are functions bounded on bounded (t,u,p) sets.

Then the boundary value problem (6.1) has at least one solution in $C^3[0,1]$.

<u>Proof</u> To prove existence of a solution in $C^3[0,1]$ we apply Theorem 6.1. To establish the a priori bounds for $(6.1)_\lambda$, let $y(t)$ be a solution to $(6.1)_\lambda$. If $\lambda = 0$ we have the unique solution $y \equiv 0$. Otherwise for $0 < \lambda \leqslant 1$, $pf(t,u,p,0) > 0$ for $|p| > M$ implies $\lambda pf(t,u,p,0) + (1 - \lambda)\, p^2 > 0$ for $|p| > M$. Thus Theorem 6.2 implies $|y'| \leqslant M$ for any solution y to $(6.1)_\lambda$. Furthermore, there exists a constant $M_1 < \infty$ independent of λ, such that $|y| \leqslant M_1$ for any solution y to $(6.1)_\lambda$. Finally we obtain a priori bounds on y'' and y'''. However, we first observe that each of the boundary conditions (i), (ii) or (iii) implies that y'' vanishes at least once on $[0,1]$. We also have
$$|f(t,u,p,q)| \leqslant A\, q^2 + B$$
where A and B denote upper bounds of $A(t,u,p)$, $B(t,u,p)$ respectively for $(t,u,p) \in [0,1] \times [-M_1,M_1] \times [-M,M]$, and so
$$|\lambda f(t,u,p,q) + (1 - \lambda)\, p| \leqslant A\, q^2 + (B + M).$$

Now each point $t \in [0,1]$ for which $y''(t) \neq 0$ belongs to an interval $[\mu, v]$ such that y'' maintains a fixed sign on $[\mu, v]$ and $y''(\mu)$ and/or $y''(v)$ is zero. Assume that $y''(\mu) = 0$ and $y'' \geq 0$ on $[\mu, v]$. Let $A_o = A, B_o = B + M$. Then

$$|\lambda f(t,y,y',y'') + (1-\lambda)y'| \leq A_o(y'')^2 + B_o,$$

and the differential equation

$$y''' = \lambda f(t,y,y',y'') + (1-\lambda)y' \quad \text{yields}$$

$$\frac{2\ A_o\ y''\ y'''}{A_o(y'')^2 + B_o} \leq 2\ A_o\ y''.$$

Integrating for μ to t we obtain

$$\mathrm{Ln}\left[\frac{A_o[y''(t)]^2 + B_o}{B_o}\right] \leq 4\ A_o\ M$$

and so

$$|y''(t)| \leq \left[\frac{B_o}{A_o}\left(e^{4A_oM} - 1\right)\right]^{1/2} \equiv M_2.$$

The other cases are treated similarly and the same bound is obtained. Thus $|y''| \leq M_2$ for each solution y to $(6.1)_\lambda$. With these bounds the differential equation yields a priori bounds independent of λ for $|y'''|$ i.e. $|y'''| \leq \max\{|f(t,y,p,q)| + |p|\} \equiv M_3$ where the maximum is computed over $[0,1] \times [-M_1, M_1] \times [-M, M] \times [-M_2, M_2]$. Thus $|y|_3 \leq K = \max\{M, M_1, M_2, M_3\}$ and the existence of a solution to (6.1) is established.

<u>Corollary 6.4</u>. Let $f : [0,1] \times R^3 \to R$ be continuous.

(a) Suppose there is a constant $M \geq 0$ such that

$$pf(t,u,p,0) \geq 0 \quad \text{for} \quad |p| > M,$$

and (t,u) in $[0,1] \times R$.

(b) Suppose that

$$|f(t,u,p,q)| \leq A(t,u,p)q^2 + B(t,u,p)$$

where $A(t,u,p)$, $B(t,u,p) \geq 0$ are functions bounded on bounded (t,u,p) sets.

Then the boundary value problem (6.1) has at least one solution in $C^3[0,1]$.

Proof Consider

$$
6.3) \qquad \begin{cases} y''' = f_n(t,y,y',y''), & t \in [0,1] \\ y \in B \end{cases}
$$

where $f_n(t,y,y',y'') = f(t,y,y',y'') + \frac{y'}{n}$ for $n = 1,2,\ldots$. Clearly $pf_n(t,u,p,0) > 0$ for $|p| > M$ and

$$|f_n(t,u,p,q)| \leqslant A(t,u,p)q^2 + (B(t,u,p) + p).$$

Apply Theorem 6.3 to (6.3) to obtain solutions y_n to (6.3) for $n = 1,2,\ldots$. In view of the previous monotonicity and bound conditions on $f_n(t,u,p,q)$ and the proof of Theorem 6.3 we also have $|y_n|_3 \leqslant K$ for some constant K independent of n. By Theorem 1.7 there is a subset N of the natural numbers and a function $y \in C^2$ so that $|y_n - y|_2 \to 0$ as $n \to \infty$ in N. If $G(t,z)$ is the Green's function for (L,B) where $Ly = y''' - y'$ we have

$$
\begin{aligned}
y_n(t) &= \int_0^1 G(t,z)[f_n(z,y_n(z),y_n'(z),y_n''(z))-y_n'(z)]dz \\
&= \int_0^1 G(t,z)[f(z,y_n(z),y_n'(z),y_n''(z))+\tfrac{1}{n}y_n'(z)-y_n'(z)]dz.
\end{aligned}
$$

Let $n \to \infty$ though N to obtain

$$y(t) = \int_0^1 G(t,z)[f(z,y(z),y'(z),y''(z)) - y'(z)]dz.$$

Thus $y \in C_B^3$ and y satisfies $y''' = f(t,y,y',y'')$.

Remark. Suppose the boundary conditions (i), (ii) or (iii) were replaced by

(iv) $y(c) = 0$, $y'(0) = 0$, $y''(1) = 0$

or

(v) $y(c) = 0$, $y'(1) = 0$, $y''(0) = 0$

where c is a fixed point of $[0,1]$. It follows easily from an analysis similar to the one above that Corollary 6.4 holds with y satisfying either (iv) or (v).

To conclude this section we examine the inhomogeneous boundary value problems,

$$6.4) \qquad \begin{cases} y''' = f(t,y,y',y''), & t \in [0,1] \\ y \in \tilde{B} \end{cases}$$

where $\tilde{B}$ denotes either the boundary conditions

(vi) $y(c) = r$, $y'(0) = \ell$, $y'(1) = T$

or

(vii) $sy(c) + dy'(c) = r$, $s \neq 0$

$-\alpha y'(0) + \beta y''(0) = \ell$, $\alpha, \beta > 0$

$ay'(1) + by''(1) = T$, $a, b > 0$.

Here c is a fixed point of $[0,1]$. We now establish the existence of a solution (6.4) with f satisfying the same hypotheses as in Theorem 6.3.

Theorem 6.5. Let $f:[0,1] \times R^3 \to R$ be continuous.

(a) Suppose there is a constant $M \geqslant 0$ such that $pf(t,u,p,0) \geqslant 0$ for $|p| > M$, and (t,u) in $[0,1] \times R$.

(b) Suppose that
$$|f(t,u,p,q)| \leqslant A(t,u,p)q^2 + B(t,u,p)$$
where $A(t,u,p)$, $B(t,u,p) \geqslant 0$ are functions bounded on bounded (t,u,p) sets.

Then the boundary value problem (6.4) has at least one solution in $C^3[0,1]$.

<u>Proof</u> Consider the family of problems

$$6.4)_\lambda \qquad \begin{cases} y''' = \lambda f(t,y,y',y''), \ 0 \leq \lambda \leq 1 \\ y \in \tilde{B}. \end{cases}$$

The existence of a solution in $C^3[0,1]$ follows immediately from Theorem 1.10 once a priori bounds independent of λ are established for solutions y to $(6.4)_\lambda$. To establish a priori bounds for $(6.4)_\lambda$, let $y(t)$ be a solution to $(6.4)_\lambda$. We assume at first that $pf(t,u,p,0) > 0$ for $|p| > M$ and (t,u) in $[0,1] \times R$. Now if $\lambda = 0$ we have a unique solution, and thus $|y'(t)| \leq L$ for some constant $L < \infty$. Otherwise for $0 < \lambda \leq 1$, $pf(t,u,p,0) > 0$ for $|p| > M$ implies $\lambda pf(t,u,p,0) > 0$ for $|p| > M$. If y satisfies (vi) it follows immediately from Theorem 6.2 that

$$|y'| \leq M_o = \max\langle M, |\ell|, |T|\rangle.$$

On the other hand if y satisfies (vii) we have

$$|y'| \leq M_1 = \max\langle M, \left|\frac{\ell}{\alpha}\right|, \left|\frac{T}{a}\right|\rangle.$$

To see this suppose $|y'(t)|$ assumes its maximum at $t = 0$. Then $y'(0)\ y''(0) \leq 0$ so

$$0 \geq y'(0)\ \beta\ y''(0) = \alpha[y'(0)]^2\left[\frac{\ell}{\alpha y'(0)} + 1\right]$$

and consequently $|y'(0)| \leq \left|\frac{\ell}{\alpha}\right|$. Likewise, $|y'(1)| \leq \left|\frac{T}{a}\right|$ if $|y'|$ achieves its maximum at $t = 1$. Thus, $|y'| \leq M_2 = \max\langle M_o, M_1, L\rangle$ for any solution y to $(6.4)_\lambda$. Furthermore there exists a constant M_3, independent of λ, such that $|y| \leq M_3$ for any solution y to $(6.4)_\lambda$. A priori bounds, independent of λ, for y'' and y''' follow exactly as in the proof of Theorem 6.3 once we observe that

$$|y''(\mu)| \leq K,$$

$K \geqslant 0$ a fixed constant independent of λ, for some point $\mu \in [0,1]$. Thus existence of a solution to (6.4) follows from Theorem 1.10.

Now assume $pf(t,u,p,0) \geqslant 0$ for $|p| > M$ and (t,u) in $[0,1] \times R$. The existence of a solution in this case follows by an argument similar to Corollary 6.4.

<u>Example</u> <u>6.1</u> (Sandwich Beam)

Beams formed by a few lamina of different materials are known as sandwich beams. In the analysis of such beams Krajcinvic [22] found that the distribution of shear deformation ψ is governed by the differential equation

$$\psi''' - k^2(x,\psi)\psi' + a(x,\psi) = 0.$$

Here $k^2 \neq 0$. For further information on k^2 and a see Krajcinvic [22].

For the case of free ends, the condition of zero shear bimoment at both ends leads to the boundary condition $\psi'(0) = \psi'(1) = 0$. Also symmetry considerations yields $\psi(1/2) = 0$. Thus we are interested in solving the boundary value problem

$$6.5) \qquad \begin{cases} \psi''' = k^2(x,\psi)\psi' - a(x,\psi), \; x \in [0,1] \\ \psi'(0) = \psi'(1) = \psi(1/2) = 0 \end{cases}$$

Now we make the following assumptions on k and a:

Suppose $k^2(x,\psi)$ and $a(x,\psi)$ are continuous functions on $[0,1] \times R$. In addition, suppose there exists a constant $L < \infty$ such that

$$\left| \frac{a(x,\psi)}{k^2(x,\psi)} \right| \leqslant L \quad \text{for} \quad (x,\psi) \in [0,1] \times R.$$

Then $\psi' f(x,\psi,\psi',0) = \psi'(k^2\psi' - a) > 0$ for $|\psi'| > L$ and $(x,\psi) \in [0,1] \times R$, and so Theorem 6.3 implies that (6.5) has at least one solution in $C^3[0,1]$.

6.3 Another Approach to Third Order Boundary Value Problems

In this section we place essentially different types of monotonicity and growth conditions on the nonlinearity f to obtain existence theorems for a wide class of third order problems. We consider problems of the form

$$\text{6.6)} \qquad \begin{cases} y''' = f(t,y,y',y''), & t \in [0,1] \\ y \in B_o \end{cases}$$

where $f:[0,1] \times R^3 \to R$ is continuous. Here B_o denotes either the boundary conditions

(viii) $y(0) = 0,\ y(1) = 0,\ y'(0) = 0$

(ix) $y(0) = 0,\ y'(0) = 0,\ y'(1) = 0$

or

(x) $-\alpha y(0) + \beta y'(0) = 0,\ \alpha,\beta > 0$

$ay(1) + by'(1) = 0,\ a,b > 0$

$y''(0) = 0.$

Remark. It should be noted here that many of the boundary conditions in section 6.2 will also be considered in this section. The behaviour of the nonlinearity f will determine which existence theorem to use.

The following theorem, although not the main result in this section, is a powerful existence theorem in its own right.

Theorem 6.6. Let $f:[0,1] \times R^3 \to R$ be continuous and $0 \leq \lambda \leq 1$. Suppose

$$|f(t,u,p,q)| \leq [A(t,u)|p| + B(t,u)][C(t,u)|q| + D(t,u)]$$

where $A(t,u)$, $B(t,u)$, $C(t,u)$, $D(t,u) \geq 0$ are functions bounded on bounded (t,u) sets. Finally, we assume that there is a constant M such that

$$|y(t)| \leq M, \quad t \in [0,1]$$

for each solution $y(t)$ to

$$6.6)_\lambda \qquad \begin{cases} y''' = \lambda f(t,y,y',y''), & t \in [0,1] \\ y \in B_o. \end{cases}$$

Then the boundary value problem (6.6) has at least one solution in $C^3[0,1]$.

<u>Proof</u> To prove existence of a solution in $C^3[0,1]$ we apply Theorem 1.9. To establish a priori bounds for $(6.6)_\lambda$, let $y(t)$ be a solution to $(6.6)_\lambda$. All that remains is to obtain a priori bounds for y', y'' and y'''. We first observe that boundary conditions (viii), (ix) or (x) imply that y'' vanishes at least once on $[0,1]$. We also have

$$|f(t,u,p,q)| \leq [A|p| + B][C|q| + D]$$

where A,B,C and D denote upper bounds of $A(t,u)$, $B(t,u)$, $C(t,u)$, $D(t,u)$ respectively for $(t,u) \in [0,1] \times [-M,M]$ and so

$$|\lambda f(t,y,y',y'')| \leq (A|y'| + B)(C|y''| + D).$$

Now each point $t \in [0,1]$ for which $y''(t) \neq 0$ belongs to an interval $[\mu,v]$ such that y'' maintains a fixed sign on $[\mu,v]$ and $y''(\mu)$ and/or $y''(v)$ is zero. Assume $y''(\mu) = 0$ and $y'' \geq 0$ on $[\mu,v]$. Then the differential equation yields

$$6.7) \qquad \frac{y'''}{Cy''+D} \leq A|y'| + B.$$

Also since $y'' \geq 0$ on $[\mu,v]$ we have y' increasing on $[\mu,v]$, so in particular $y'(s) \geq y'(\mu)$ for $s \in [\mu,v]$. At this stage of the proof the argument breaks up into two cases, $y'(\mu) \geq 0$ and $y'(\mu) < 0$.

Assume at first $y'(\mu) \geqslant 0$, and so $y'(s) \geqslant 0$ for $s \in [\mu, v]$. It follows from (6.7) that

$$\frac{Cy'''}{Cy''+D} \leqslant ACy' + BC$$

and so integrating from μ to t we obtain

$$|y''(t)| \leqslant \frac{D}{C}\,[\exp(2ACM + BC) - 1] \equiv M_o.$$

On the other hand assume $y'(\mu) < 0$. Again the argument breaks up into two subcases, $y'(s) \leqslant 0$ for $s \in [\mu, v]$ or there exists $\zeta \in (\mu, t)$ such that $y'(\zeta) = 0$ and $y'(s) > 0$ for $s \in (\zeta, v]$. Suppose at first $y'(s) \leqslant 0$ for $s \in [\mu, v]$, then (6.7) implies

$$\frac{Cy'''}{Cy''+D} \leqslant - ACy' + BC$$

and so integrating from μ to t yields

$$|y''(t)| \leqslant M_o,$$

as before. Finally suppose there exists $\zeta \in (\mu, t)$ such that $y'(\zeta) = 0$ and $y'(s) > 0$ for $s \in (\zeta, v]$, then for $\eta \in [\mu, \zeta]$, (6.7) implies

$$\frac{Cy'''}{Cy''+D} \leqslant - ACy' + BC,$$

which yields $|y''(\zeta)| \leqslant M_o$. So for $s \in [\zeta, v]$, (6.7) again implies

$$\frac{Cy'''}{Cy''+D} \leqslant ACy' + BC,$$

and thus

$$|y''(t)| \leqslant \frac{[CM_o + D]\exp(2ACM+BC) - D}{C} \equiv M_1.$$

Thus $|y''(t)| \leqslant M_1$. The other cases are treated similarly and the same bound $M_1 = \max\langle M_o, M_1\rangle$ is obtained. Thus $|y''| \leq M_1$ for each solution y to $(6.6)_\lambda$. Now if y satisfies (viii) then

$$|y'(t)| = \left|\int_0^t y''(z)dz\right| \leq M_1,$$

while if y satisfies (ix)

$$|y'(t)| = \left|\int_t^1 y''(z)dz\right| \leq M_1.$$

Finally if y satisfies (x)

$$|y'(t)| \leq \left|\int_0^t y''(z)dz\right| + |y'(0)| \leq M_1 + \frac{\alpha M}{\beta} \equiv M_2.$$

Thus $|y'| \leq M_2$ for each solution y to $(6.6)_\lambda$. With these bounds the differential equation yields a priori bounds independent of λ for $|y'''|$ i.e. $|y'''| \leq \max\{|f(t,u,p,q)|\} \equiv M_3$ where the maximum is computed over $[0,1] \times [-M,M] \times [-M_2,M_2] \times [-M_1,M_1]$. Thus $|y|_3 \leq K = \max\{M,M_1,M_2,M_3\}$ and the existence of a solution to (6.6) is established.

For notational purposes, let $(6.6)_{(viii)}$ denote the boundary value problem $y''' = f(t,y,y',y'')$, $t\in[0,1]$, with y satisfying (viii). Similarly, define $(6.6)_{(ix)}$ and $(6.6)_{(x)}$. Next sufficent conditions on f are given which imply a priori bounds on any solution $y(t)$ to $(6.6)_{(viii)}$, $(6.6)_{(ix)}$ or $(6.6)_{(x)}$. Suppose $y(t)$ is a solution to (6.6) and $[y(t)]^2$ has a maximum at $t_o \in (0,1)$. Then $y'(t_o) = 0$ and $y(t_o)\, y''(t_o) \leq 0$.

<u>Theorem 6.7</u>. Suppose there is a constant $M \geq 0$ such that

$$\int_0^t u'(z)[f(z,u(z),u'(z),u''(z)) + L(u'(z))^n u''(z)]dz > 0$$

for $|u(t)| > M$, where L and $n > -2$ are constants, with $u \in C^2[0,1]$ and $u'(0) = 0$.

Then any solution y to $(6.6)_{(viii)}$ or $(6.6)_{(ix)}$ satisfies

$$|y(t)| \leqslant M \quad \text{for} \quad t \in [0,1].$$

<u>Proof</u> Suppose $|y|$ achieves a positive maximum at $t_o \in (0,1)$, then $y'(t_o) = 0$. Assume $|y(t_o)| > M$, and so

$$\int_o^{t_o} \Big[y'(z)y'''(z) + L(y'(z))^{n+1}y''(z)\Big]dz > 0.$$

Integration by parts together with $y'(t_o) = 0$ yields

$$-\int_0^{t_o} (y''(z))^2\, dz > 0,$$

a contradiction. Thus, $|y(t_o)| \leqslant M$.

At this stage we divide the proof into two cases. Suppose first y is a solution to $(6.6)_{(viii)}$. If $|y|$ assumes its maximum value at either $t = 0$ or $t = 1$ then trivially $|y(t)| \leqslant M$ for $t \in [0,1]$. So conclusion of theorem follows for $(6.6)_{(viii)}$. Now suppose y is a solution to $(6.6)_{(ix)}$. If y assumes its maximum value at $t = 0$ then trivially $|y(t)| \leqslant M$ for $t \in [0,1]$. On the other hand suppose $|y|$ achieves its maximum value at $t = 1$. Suppose $|y(1)| > M$, and so

$$\int_0^1 \Big[y'(z)y'''(z) + L(y'(z))^{n+1}y''(z)\Big]dz > 0$$

which yields

$$-\int_0^1 [y''(z)]^2 dz > 0,$$

a contradiction. Thus $|y(1)| \leq M$ and the conclusion of the theorem follows for $(6.6)_{(ix)}$.

<u>Remark</u>. M is independent of L in Theorem 6.7.

An analogous theorem holds for $(6.6)_{(x)}$.

<u>Theorem 6.8</u>. Suppose there is a constant $M \geq 0$ such that

$$\int_0^t u'(z)\Big[f(z,u(z),u'(z),u''(z)) + L(u'(z))^n u''(z)\Big]dz > 0$$

for $|u(t)| > M$, where n is an even integer greater than or equal to zero and $L \geq 0$ is a constant, with $u \in C^2[0,1]$ and $u''(0) = 0$

Then any solution y to $(6.6)_{(x)}$ satisfies

$$|y(t)| \leq M \quad \text{for} \quad t \in [0,1].$$

<u>Proof</u> Suppose $|y|$ achieves a positive maximum at $t_o \in (0,1)$ and assume $|y(t_o)| > M$. Then $y'(t_o) = 0$ and

$$\int_0^{t_o} \Big[y'(z)y'''(z) + L(y'(z))^{n+1}y''(z)\Big]dz > 0$$

yields

$$-\int_0^{t_o} [y''(z)]^2 dz - L\,\frac{(y'(0))^{n+2}}{n+2} > 0,$$

a contradiction. Thus $|y(t_o)| \leq M$.

On the other hand $|y|$ cannot have a nontrivial maximum at 0 or 1. For suppose the maximum of $|y|$ occurs at 0. Then $y(0)\ y'(0) \leq 0$. However, from (x) $y(0)\ y'(0) = \frac{\beta}{\alpha}[y(0)]^2 > 0$, a contradiction. A similar argument works for the case $t = 1$. Thus

$$|y(t)| \leqslant M \quad \text{for} \quad t \in [0,1].$$

We are now in a position to prove our main existence theorems for this section.

<u>Theorem</u> <u>6.8</u>. Let $f:[0,1] \times R^3 \to R$ be continuous.

(a) Suppose there is a constant $M \geqslant 0$ such that

$$\int_0^t u'(z)[f(z,u(z),u'(z),u''(z)) + L(u'(z))^n u''(z)]dz > 0$$

for $|u(t)| > M$, where L and $n > -2$ are constants, with $u \in C^2[0,1]$ and $u'(0) = 0$.

(b) Suppose

$$|f(t,u,p,q)| \leqslant [A(t,u)|p|+B(t,u)][C(t,u)|q|+D(t,u)]$$

where $A(t,u)$, $B(t,u)$, $C(t,u)$, $D(t,u) \geqslant 0$ are functions bounded on bounded (t,u) sets.

Then the boundary value problem $(6.6)_{(viii)}$ and $(6.6)_{(ix)}$ have at least one solution in $C^3[0,1]$.

<u>Proof</u> To prove existence of a solution in $C^3[0,1]$ we apply Theorem 6.6. We need establish a priori bounds for any solution $y(t)$ to $(6.6)_\lambda$. Now if $\lambda = 0$ we have the unique solution $y \equiv 0$. Otherwise for $0 < \lambda \leqslant 1$

$$\int_0^t u'(z)[f(z,u(z),u'(z),u''(z))+L(u'(z))^n u''(z)]dz>0$$

for $|u(t)| < M$ implies

$$\int_0^t u'(z)[\lambda f(z,u(z),u'(z),u''(z))+\lambda L(u'(z))^n u''(z)]dz>0$$

for $|u(t)|>M$ and so Theorem 6.7 together with its remark implies $|y| \leq M$ for any solution y to $(6.6)_\lambda$. Hence the existence of a solution to $(6.6)_{(viii)}$ and $(6.6)_{(ix)}$ is established.

<u>Corollary 6.10</u>. Let $f:[0,1] \times R^3 \to R$ be continuous.

(a) Suppose there is a constant $M \geq 0$ such that

$$\int_0^t u'(z)[f(z,u(z),u'(z),u''(z))+L[u'(z)]^n u''(z)]dz \geq 0$$

for $|u(t)| > M$ where L and $n > -2$ are constants, with $u \in C^2[0,1]$ and $u'(0) = 0$.

(b) Suppose

$$|f(t,u,p,q)| \leq [A(t,u)|p|+B(t,u)][C(t,u)|q|+D(t,u)]$$

where $A(t,u)$, $B(t,u)$, $C(t,u)$, $D(t,u) \geq 0$ are functions bounded on bounded (t,u) sets.

Then the boundary value problem $(6.6)_{(viii)}$ and $(6.6)_{(ix)}$ have at least one solution in $C^3[0,1]$.

<u>Proof</u> Let us consider

$$6.8) \qquad \begin{cases} y''' = f_n(t,y,y',y'') \\ y \in (viii) \quad \text{or} \quad (ix) \end{cases}$$

where $f_n(t,y,y',y'') = f(t,y,y',y'') + \frac{y}{n}$ for $n = 1,2,\ldots$. Clearly

$$\int_0^t y'(z)[f_n(z,y(z),y'(z),y''(z))+L(y'(z))^n y''(z)]dz$$

$$= \int_0^t y'(z)[f(z,y(z),y'(z)y''(z))+L(y'(z))^n y''(z)]dz+\frac{1}{n}\,\frac{y^2(t)}{2}$$

> 0

for $|y(t)| > M$ since $y(0) = 0$.
Thus Theorem 6.7 implies $|y| \leq M$ for any solution y_n to (6.8) and $n = 1,2,\ldots$. Also

$$|f_n(t,y,y',y'')|$$
$$\leq [A(t,y)|y'|+B(t,y)][C(t,y)|y''|+D(t,y)]+M.$$

Now we can apply Theorem 6.9 to (6.8): If y_n is a solution to (6.8) for $n = 1,2,\ldots$ we have $|y_n|_3 \leq K$ for some constant K independent of n. By Theorem 1.7 there is a subset N of the natural numbers and a function $y \in C^2[0,1]$ so that $|y_n-y|_2 \to 0$ as $n \to \infty$ in N. If $G(t,z)$ is the Green's function for (L,B_o) where $Ly = y'''$ and B_o denotes the boundary conditions (viii) or (ix) then

$$y_n(t) = \int_0^1 G(t,z)f_n(z,y_n(z),y_n'(z),y_n''(z))dz.$$

Let $n \to \infty$ through N to obtain

$$y(t) = \int_0^1 G(t,z)f(z,y(z),y'(z),y''(z))dz.$$

Thus $y \in C^3_{B_o}$ and y satisfies $y''' = f(t,y,y',y'')$.

We can obtain a similar result to Theorem 6.9 for $(6.6)_{(x)}$.

<u>Theorem 6.11</u>. Let $f:[0,1] \times R^3 \to R$ be continuous.

(a) Suppose there is a constant $M \geq 0$ such that

$$\int_0^t u'(z)[f(z,u(z),u'(z),u''(z))+L(u'(z))^n u''(z)]dz > 0$$

for $|u(t)| > M$, where n is an even integer greater than or equal to zero and $L \geq 0$ is a constant, with $u \in C^2[0,1]$ and $u''(0) = 0$.

(b) Suppose

$$|f(t,u,p,q)| \leqslant [A(t,u)|p|+B(t,u)][C(t,u)|q|+D(t,u)]$$

where $A(t,u)$, $B(t,u)$, $C(t,u)$, $D(t,u) \geqslant 0$ are functions bounded on bounded (t,u) sets.

Then the boundary value problem $(6.6)_{(x)}$ has at least one solution in $C^3[0,1]$.

<u>Remark</u>. We can obtain similar results to those in Theorem's 6.9 and 6.11 if the boundary conditions (viii), (ix) or (x) are replaced by either

(xi) $y(0) = 0,\ y(1) = 0,\ y''(0) = 0$

(xii) $y(0) = 0,\ y'(1) = 0,\ y''(0) = 0$

(xiii) $y(1) = 1,\ y'(0) = 0,\ y'(1) = 0$

(xiv) $y(0) = 0,\ y(1) = 0,\ y'(1) = 0$

(xv) $y(0) = 0,\ y(1) = 1,\ y''(1) = 0$

(xvi) $y(1) = 0,\ y'(0) = 0,\ y''(1) = 0$

or

(xvii) $-\alpha y(0) + \beta y'(0) = 0,\ \alpha,\beta > 0$
$ay(1) + by'(1) = 0,\ a,b > 0$
$y''(1) = 0.$

An example of this is the following:

<u>Theorem 6.12</u>. Let $f:[0,1] \times R^3 \to R$ be continuous.

(a) Suppose there is a constant $M \geqslant 0$ such that

$$\int_t^1 u'(z)[f(z,u(z),u'(z)u''(z))+L(u'(z))^n u''(z)]dz > 0$$

for $|u(t)| > M$, where n is an even integer greater than or equal to zero and $L \leqslant 0$ is a constant, with $u \in C^2[0,1]$ and $u''(1) = 0$.

(b) Suppose

$$|f(t,u,p,q)| \leqslant [A(t,u)|p|+B(t,u)][C(t,u)|q|+D(t,u)]$$

where $A(t,u)$, $B(t,u)$, $C(t,u)$, $D(t,u) \geqslant 0$ are functions bounded on bounded (t,u) sets.

Then the boundary value problem

$$\begin{cases} y''' = f(t,y,y',y''), & t \in [0,1] \\ y \in (xvii) \end{cases}$$

has at least one solution in $C^3[0,1]$.

The following example illustrates the ideas and results of this section.

<u>Example</u> <u>6.2</u> Consider the boundary value problem

$$\text{BVP)} \qquad \begin{cases} y'''(t) = A_o + B_o[y(t)]^p + C_o[y(t)]^m y'(t) + D_o y''(t) \\ \qquad + E_o y'(t)y''(t), t \in [0,1] \\ y(0) = y(1) = y'(0) = 0 \end{cases}$$

where A_o, $B_o > 0$, $C_o \geqslant 0$, D_o, E_o are constants with $m \geqslant 0$ even and $p > 0$ odd.

We will now show that (BVP) has a solution in $C^3[0,1]$ via Theorem 6.8. Now if $f(t,y,y',y'') = A_o + B_o\, y^p + C_o y^m\, y' + D_o\, y'' + E_o\, y'\, y''$ and $L = -E_o$, $n = 1$, $\tilde{L} = -D_o$, $\tilde{n} = 0$ we have

$$\int_0^t y'(z)[f(z,y(z),y',y'') + L(y'(z))^n y''(z) + \tilde{L}(y'(z))^{\tilde{n}} y''(z)]dz$$

$$= \int_0^t \left[A_o y'(z) + B_o[y(z)]^p y'(z) + C_o[y(z)]^m[y'(z)]^2\right]dz$$

$$\geqslant \int_0^t [A_o y'(z) + B_o(y(z))^p y'(z)]dz \quad \text{since} \quad y'(0) = 0$$

$$= y(t)\left[A_o + \frac{B_o}{p+1}[y(t)]^p\right] \quad \text{since} \quad y(0) = 0$$

> 0 for $|y(t)| > \left[\left|\frac{-(p+1)A_o}{B_o}\right|\right]^{1/p}$

Finally it is clear that we can find constants A,B,C,D such that

$$|f(t,y,y',y'')| \leq (A|y'|+B)(C|y''|+D)$$

for (t,y) in a bounded set.

Hence all conditions in Theorem 6.8 are satisfied so (BVP) has at least one solution in $C^3[0,1]$.

To conclude this chapter we examine the inhomogeneous boundary value problem

6.9) $$\begin{cases} y''' = f(t,y,y',y''), \ t \in [0,1] \\ y(0) = r, \ y(1) = s, \ y'(0) = \ell \end{cases}$$

or

6.10) $$\begin{cases} y''' = f(t,y,y',y''), \ t \in [0,1] \\ y(0) = r, \ y(1) = s, \ y''(0) = \ell \end{cases}$$

and establish the existence of a solution (6.9), (6.10) in $C^3[0,1]$ under essentially the same hypothesis on f used in Theorem 6.9.

<u>Theorem</u> <u>6.13</u>. Let $f:[0,1] \times R^3 \to R$ be continuous.

(a) Suppose there is a constant $M \geq 0$ such that

$$\int_0^t u'(z)[f(z,u(z)u'(z),u''(z))+L(u'(z))^n u''(z)]dz > 0$$

for $|u(t)| > M$, where L and $n > -2$ are constants with $u \in C^2[0,1]$ and $u'(0) = 0$.

(b) Suppose

$$|f(t,u,p,q)| \leq [A(t,u)|p|+B(t,u)][C(t,u)|q|+D(t,u)]$$

where $A(t,u)$, $B(t,u)$, $C(t,u)$, $D(t,u) \geqslant 0$ are functions bounded on bounded (t,u) sets.

Then the boundary value problem (6.9) has at least one solution in $C^3[0,1]$.

<u>Proof</u> Consider the family of problems

$$\text{6.9)}_\lambda \qquad \begin{cases} y''' = \lambda f(t,y,y',y''), \; 0 \leqslant \lambda \leqslant 1 \\ y(0) = r, \; y(1) = s, \; y'(0) = \ell. \end{cases}$$

The existence of a solution in $C^3[0,1]$ follows immediately from Theorem 1.10 once a priori bounds independent of λ are established for solutions y to $(6.9)_\lambda$. To establish a priori bounds for $(6.9)_\lambda$, let $y(t)$ be a solution to $(6.9)_\lambda$. Now if $\lambda = 0$ we have a unique solution and thus $|y(t)| \leqslant M_o$, for some constant M_o. Otherwise for $0 < \lambda \leqslant 1$, let $w = y - \ell t$, so $w(0) = r$, $w(1) = s - \ell$, $w'(0) = 0$. Now

$$\int_0^t w'(z)[f(z,w(z),w'(z),w''(z))+L(w'(z))^n w''(z)]dz > 0$$

for $|w(t)| > M$ implies

$$\int_0^t w'(z)[\lambda f(z,w(z),w'(z),w''(z)+\lambda L(w'(z))^n w''(z)]dz > 0$$

for $|w(t)| > M$. It follows from Theorem 6.7 and its remark that

$$|w(t)| \leqslant \max\langle M,|r|,|s-\ell|\rangle = \tilde{K} \quad \text{for} \quad t \in [0,1].$$

Thus $|y(t)| \leqslant M_1 \equiv \max\langle \tilde{K}+|\ell|,M_o\rangle$ for any solution y to $(6.9)_\lambda$, $0 \leqslant \lambda \leqslant 1$. A priori bounds independent of λ for y',y'',y''' will follow from a slight modifictation of the proof in Theorem 6.6 once we observe that $|y''(\mu)| \leqslant K$, $K \geqslant 0$ a fixed constant independent of λ, for some $\mu \in [0,1]$. We accomplish this by letting $v(t) = y(t) - (1 - t^2)r - t^2 s + \ell t^2$ and noticing that $v(0) = 0$, $v(1) = \ell$, $v'(0) = \ell$. Hence by the Mean Value Theorem there exists $\mu \in (0,1)$ such that $v''(\mu) = 0$ i.e. $y''(\mu) = 2s - 2\ell - 2r$. The existence of a solution to (6.9) follows from Theorem 1.10.

We can obtain a corresponding existence theorem for (6.10):

<u>Theorem</u> <u>6.14</u>. Let $f:[0,1] \times R^3 \to R$ be continuous.

(a) Suppose there is a constant $M \geq 0$ such that

$$\int_0^t u'(z)[f(z,u(z),u'(z),u''(z))+L(u'(z))^n u''(z)]dz > 0$$

for $|u(t)| > M$, where n is an even integer greater than or equal to zero and $L \geq 0$ is a constant, with $u \in C^2[0,1]$ and $u''(0) = 0$.

(b) Suppose

$$|f(t,u,p,q)| \leq [A(t,u)|p|+B(t,u)][C(t,u)|q|+D(t,u)]$$

where $A(t,u)$, $B(t,u)$, $C(t,u)$, $D(t,u) \geq 0$ are functions bounded on bounded (t,u) sets.

Then the boundary value problem (6.10) has at least one solution in $C^3[0,1]$.

VII. Nonlinear Differential Equations in Hilbert Spaces

7.1 Introduction

The theory of nonlinear differential equations in abstract spaces became popular in the 1970's and is still being studied in great depth. For a detailed account of the subject see Deimling [10], Lakshmikantham and Leela [24] and Martin [25]. In this chapter we present a new approach via the Topological Transversality Theorem, to studying problems of the form

$$\text{7.1)} \qquad \begin{cases} y' = f(t,y), & t \in [0,T] \\ y(0) = y_o. \end{cases}$$

Here y takes values in a real Hilbert space $(H, \|\cdot\|)$, $y_o \in H$ and $f:[0,T] \times H \to H$ is continuous.

For notational purposes let $C^1([0,T],H)$ denote the space of continuously differentiable functions g on $[0,T]$. Now $C^1([0,T],H)$ with norm

$$\|g\|_1 = \max\left\{ \sup_{t\in[0,T]} \|y(t)\|, \sup_{t\in[0,T]} \|y'(t)\| \right\}$$

$$\equiv \max\left\{ \|y\|_o, \|y'\|_o \right\}$$

is a Banach space. Similarly we define $C([0,T],H)$. Finally, by a solution to (7.1) we mean a function $y \in C^1([0,T],H)$ together with y satisfying $y' = f(t,y), t\in[0,T]$, and $y(0) = y_o$.

Unlike the finite dimensional case continuity assumptions on f alone will not guarantee even local existence; see Banas and Goebel [2]. In this chapter by placing compactness conditions on f we obtain, with a restriction on T which depends on the nonlinearity f, solutions to (7.1) in $C^1([0,T],H)$.

7.2 Preliminary Results

We begin with some standard theorems on the calculus of functions from an interval into a real Hilbert space; see Martin [25], Barbu [3] and Shilov [30] for details. Suppose for the remainder of this section that H is a real Hilbert space $(H,\langle\cdot,\cdot\rangle)$ and J is a compact interval in R.

<u>Theorem 7.1</u>. Suppose f is a differentiable function from J into H and $f'(t) = 0$ for all $t \in J$. Then f is constant on J.

<u>Theorem 7.2</u>. Suppose f is a differentiable function from J into H. Then

$$\frac{d}{dt}\langle f(t),f(t)\rangle = 2\langle f'(t),f(t)\rangle.$$

<u>Theorem 7.3</u>. Suppose $J = [a,b]$ and $f(u)$ is a continuous function from J into H. Also let $u = u(t)$ be a continuously differentiable function on $\alpha \leqslant t \leqslant \beta$ where $u(\alpha) = a$ and $u(\beta) = b$. Then

$$\int_a^b f(u)du = \int_\alpha^\beta f(u(t))u'(t)dt.$$

To obtain our existence theorems in the following section we need a more general version of the Arzela Ascoli Theorem.

<u>Theorem 7.4</u>. Suppose M is a subset of $C(J,H)$. Then M is relatively compact in $C(J,H)$ (i.e. $\bar{M}$ is a compact subset of $C(J,H)$) if and only if M is bounded, equicontinuous and the set $\{f(t):f \in M\}$ is relatively compact for each $t \in J$.

7.3 Initial Value Problems in Hilbert Spaces

We begin, as usual, by examining the homogeneous first order initial value problem

$$7.2) \qquad \begin{cases} y' = f(t,y), & t \in [0,T] \\ y(0) = 0 \end{cases}$$

where y takes values in a real Hilbert space $(H, \langle \cdot , \cdot \rangle)$ and $f:[0,T] \times H \to H$ is continuous. Let $\| \cdot \|^2 = \langle \cdot , \cdot \rangle$.

Now the Topological Transversality Theorem and the Arzela Asocoli Theorem are used to extend Theorem 1.9 for initial value problems in Hilbert spaces.

<u>Theorem</u> <u>7.5</u>. Let $f:[0,T] \times H \to H$ be continuous and $0 \leqslant \lambda \leqslant 1$ Suppose in addition f satisfies the following:

$$7.3) \qquad \begin{cases} \text{There is a continuous function } \psi:[0,\infty) \to (0,\infty) \\ \text{such that } \|f(t,y)\| \leqslant \psi(\|y\|). \end{cases}$$

7.4) f is completely continuous on $[0,T] \times H$.

$$7.5) \qquad \begin{cases} \text{For } t,s \in [0,T] \text{ and } \Omega \text{ a bounded subset} \\ \text{of } C^1([0,T],H), \text{ there exists a constant} \\ A \geqslant 0 \text{ (which can depend on } \Omega) \text{ such that} \\ \qquad \|f(t,u(t)) - f(s,u(s))\| \leqslant A|t-s| \\ \text{for all } u \in \Omega. \end{cases}$$

Finally suppose there is a constant K such that $\|y\|_1 \leqslant K$ for each solution $y(t)$ to

$$7.2)_\lambda \qquad \begin{cases} y' = \lambda f(t,y), & t \in [0,T] \\ y(0) = 0. \end{cases}$$

Then the boundary value problem (7.2) has at least one solution in $C^1([0,T],H)$.

<u>Proof</u> As before let

$C_B^1([0,T],H) = \{u \in C^1([0,T],H): \ u(0) = 0\}$. Also let $\bar{V} = \{u \in C_B^1([0,T],H): \ \|u\|_1 \leq K+1\}$ and define $F_\lambda : C_B^1([0,T],H) \to C([0,T],H)$, $0 \leq \lambda \leq 1$, by $F_\lambda[u](t) = \lambda f(t,v(t))$. Now assumptions (7.3), (7.4) and (7.5) together with Theorem 7.4 imply that F_λ is completely continuous. To see this let Ω be a bounded subset of $C^1([0,T],H)$, then for $u \in \Omega$ $\|F_\lambda u\| = \|\lambda f(t,u)\| \leq \psi(\|u\|) \equiv M_o$ where $M_o < \infty$ is a constant. Clearly from (7.5) $F_\lambda(\Omega)$ is equicontinuous and we have also for each t $[0,T]$, $F(\Omega(t)) = \{f(t,u(t));\ u \in \Omega\}$ which is relatively compact in H since f is completely continuous.

Finally we define $L : C_B^1([0,T],H) \to C([0,T],H)$ by $Ly = y'$. It follows from Theorem 1.6 that L^{-1} is a bounded linear operator. Thus $H_\lambda = L^{-1}F_\lambda$ defines a homotopy $H_\lambda : \bar{V} \to C_B^1([0,T],H)$. It is clear that the fixed points of H_λ are precisely the solutions to $(7.2)_\lambda$. Moreover the complete continuity of F_λ together with the continuity of L^{-1} imply that the homotopy H_λ is compact. Now H_o is essential so Theorem 1.5 implies that H_1 is essential. Thus (7.2) has a solution.

<u>Remark</u>. If we replace the Hilbert space H with a Banach space $\tilde{B}$ then again Theorem 7.5 holds with $\tilde{B}$ replacing H.

In view of Theorem 7.5 we obtain immediately:

<u>Theorem 7.5</u>. Suppose $f : [0,T] \times H \to H$ is continuous and satisfies (7.3), (7.4) and (7.5). Then the initial value problem (7.2) has a solution in $C^1([0,T],H)$ for each $T < \int_0^\infty \frac{du}{\psi(u)}$.

<u>Proof</u> To prove existence of a solution in $C^1([0,T],H)$ we apply Theorem 7.5. To establish a priori bounds for

$(7.2)_\lambda$, let $y(t)$ be a solution to $(7.2)_\lambda$. Then

$$\|y'\| = \|\lambda f(t,y)\| \leqslant \psi(\|y\|).$$

Now if $\|y(t)\| \neq 0$ we have from Theorem 7.2 and the Cauchy Schwartz inequality that

$$\|y\|' = \frac{\langle y', y\rangle}{\|y\|} \leqslant \|y'\|$$

and the inequality above yields

$$\|y\|' \leqslant \psi(\|y\|)$$

at any point t where $\|y(t)\| \neq 0$. Suppose $\|y(t)\| \neq 0$ for some point $t \in [0,T]$. Then since $y(0) = 0$ there is an interval $[a,t]$ in $[0,T]$ such that $\|y(s)\| > 0$ on $a < s \leqslant t$ and $\|y(a)\| = 0$. Then the previous inequality implies

$$\int_a^t \frac{\|y(s)\|'}{\psi(\|y(s)\|)}\, ds \leqslant t - a.$$

$$\text{So,} \quad \int_0^{\|y(t)\|} \frac{du}{\psi(u)} \leqslant T < \int_0^\infty \frac{du}{\psi(u)} .$$

This inequality implies there is a constant M_1 such that $\|y\|_o \leqslant M_1$. Also $(7.2)_\lambda$ and (7.3) implies $\|y'(t)\| \leqslant \max\limits_{0 \leqslant u \leqslant M_1} \psi(u) \equiv M_2$ for some constant M_2. So $\|y\|_1 \leqslant K = \max\{M_1, M_2\}$ and the existence of a solution is established.

Theorem 7.5 also holds for the inhomogeneous initial condition $y(0) = y_o \in H$. In fact Theorem 1.10 and trivial adjustments in the above proof yield:

<u>Theorem</u> <u>7.7</u>. Suppose $f:[0,T] \times H \to H$ is continuous and satisfies (7.3), (7.4) and (7.5). Then the initial value problem

$$\begin{cases} y' = f(t,y), \ t \in [0,T] \\ y(0) = y_o \in H \end{cases}$$

has a solution in $C^1([0,T],H)$ for each

$$T < \int_{\|y_o\|}^{\infty} \frac{du}{\psi(u)} .$$

<u>Example</u> <u>7.1</u> The techniques above may be applied to integro-differential equations of the form

$$7.6) \quad \begin{cases} \frac{\partial}{\partial t} y(t,s) = \int_o^T g(t,s,r,y(t,r))dr; \ t,s \in [0,T] \\ y(0,s) = \mu(s) \end{cases}$$

where $\mu:[0,T] \to R$ is continuous.

Let $H = L^2([0,T],R)$, with the usual inner product and define the mapping B from $[0,T] \times H$ into H by

$$[B(t,u)](s) = \int_0^T g(t,s,r,u(r))dr \quad \text{for all}$$

$(t,s,u) \in [0,T] \times [0,T] \times E$ where $E \subset H$.

We begin by examining the initial value problem

$$7.7) \quad \begin{cases} u' = B(t,u(t)), \ t \in [0,T] \\ u(0) = \mu \end{cases}$$

where $B:[0,T] \times H \to H$.

Various conditions on g insuring the continuity and complete continuity of B from $[0,T] \times H$ into H may be found in Krasnoselski [23]. We also assume g satisfies certain growth conditions so that

$$\|B(t,u)\|_{L^2} \leq \psi(\|u\|_{L^2})$$

where $\psi:[0,\infty) \to (0,\infty)$ is continuous. Now assume

$T < \int_{\|\mu\|_{L^2}}^{\infty} \frac{du}{\psi(u)}$. Finally suppose conditions are put on g so that, for $t, t' \in [0,T]$ and Ω a bounded subset of $C^1([0,T],H)$, there exists a constant $A \geqslant 0$ such that

$$\|B(t,u(t)) - B(t',u(t'))\|_{L^2} \leqslant A|t-t'|$$

for all $u \in \Omega$.

Then Theorem 7.7 implies that (7.7) has a solution on $[0,T]$. Suppose $u(t)$ is a solution to (7.7) on $[0,T]$, then one sees that if $y(t,s) = [u(t)](s)$ for all $t \in [0,T]$ and $s \in (0,T]$, then $y(0,\alpha) = \mu(\alpha)$ and y is a solution to (7.6). To see this let $[u(t)](s) \equiv v(s)$, so

$$\frac{\partial}{\partial t} y(t,s) = B(t,v)(s) = \int_0^T g(t,s,r,v(r))dr$$

$$= \int_0^T g(t,s,r,y(t,r))dr.$$

BIBLIOGRAPHY

1. P.B. Bailey, L.F. Shampine and P.E. Waltman: Nonlinear Two Point Boundary Value Problems, Academic Press, New York and London, 1968.

2. J. Banas and K. Goebel: Measures of Noncompactness in Banach Spaces, Marcel Dekker Inc., New York and Basel, 1980.

3. V. Barbu: Nonlinear Semigroups and Differential Equations in Banach Spaces, Noordhoff International Publishing, 1976.

4. T.A. Botts and E.J. McShane: Real Analysis, D. Van Nostrand Co. Inc., 1959.

5. E.M. Buturla, P.E. Cottrell, B.M. Grossman and K.A. Salsburg: Finite-Elements Analysis of Semiconductor Devices, The FIELDAY program IBM J. Res. Dev., 24 (1981), 218-231.

6. Choong-Ki Kim: The physics of charge coupled devices in change coupled devices and systems, edited by M.J. Howes and D.V. Morgan, Wiley-Interscience, New York, 1979.

7. J.B. Conway: Functions of one Complex Variable, Springer-Verlag, New York - Heidelberg and Berlin, 1978.

8. C. Corduneanu: Principles of Differential and Integral Equations, Chelsea Pub. Co., New York, 1977.