FLAT PLATES OF PLYWOOD
UNDER UNIFORM OR
CONCENTRATED LOADS

March 1942

INFORMATION REVIEWED
AND REAFFIRMED
1962

THIS REPORT IS ONE OF A SERIES ISSUED
TO AID THE NATION'S WAR PROGRAM

No. 1312

UNITED STATES DEPARTMENT OF AGRICULTURE
FOREST SERVICE
FOREST PRODUCTS LABORATORY
Madison 5, Wisconsin

In Cooperation with the University of Wisconsin
FLAT PLATES OF PLYWOOD UNDER UNIFORM OR CONCENTRATED LOADS*

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*This mimeograph is one of a series of progress reports issued by the Forest Products Laboratory to aid the Nation's defense effort. Results here reported are preliminary and may be revised as additional data become available.

Mimeo. No. 1312
Introduction

This report presents the results of a study made by the Forest Products Laboratory of the behavior of flat plates of plywood under uniform or concentrated loads.

The information concerning the elastic properties of plywood developed in this study will, it is hoped, be useful in the treatment of further problems such as the buckling of flat and curved plywood panels.

After a discussion of the elastic properties of wood the remainder of the report is divided into two main portions, one dealing with plates under such loads that the deflections are small, the other with plates under such loads that the deflections are large. For plates with small deflections, direct stresses ("membrane stresses") throughout the thickness of the plate due to the deformation of the middle surface, are negligible and bending stresses only need to be considered. These consist in tensile stresses on one side of the plate and compressive stresses on the other side. For a plate to be considered in the class of those with small deflections the deflection must usually be less than the thickness of the plate, in certain cases less than one-half of this thickness. For larger loads and consequent larger deflections direct stresses are developed to such an extent that they cannot be neglected. The loads are carried partly by such stresses and partly by bending stresses. In such cases the linear relationship between load and deflection, which holds for small loads, is no longer maintained.

To the reader who does not wish to follow the mathematical analysis it should be pointed out that the majority of the results are presented in forms which do not presuppose for their application a knowledge of their theoretical derivation. It is suggested that such a reader should first become familiar with the significance of the two mean Young's moduli in bending (Section 2), after having read Section 1 on the elastic behavior of wood. The curves of figures 10, 18, and 26 can then be used to determine the maximum deflections of plates with small deflections under uniformly distributed loads or concentrated loads. Correspondingly simple means of determining maximum bending stresses have not been worked out. However, for uniformly loaded plates whose lengths are greater than moderate multiples of their breadths, the stresses in the central portions can be easily approximated by calculating the stresses in similarly loaded infinitely long strips to which essentially the simple beam formulas are applicable. The details of the procedure are explained in Section 3. For stresses in the vicinity of concentrated loads the methods of Sections 5 and 6 are available. The tables for bending moments have been calculated only for three-ply plates. Formulas requiring considerable computation are given by which the moments can be calculated for other types of plates. It appears that bending moments in the vicinity of the loads can in most cases be calculated with sufficient accuracy by considering the plates to be infinitely long.

For plates with large deflections the approximate formulas of Sections 9 and 10 for long narrow plates can be used for uniformly
distributed loads to find both the maximum deflection and stress in a plate whose length exceeds its breadth by a moderate amount. In Sections 11 and 12 approximate formulas are given for the maximum deflection of a plate of smaller length-to-breadth ratio with large deflection under a uniformly distributed load. The tables presented for different types of plates enable one to estimate the ratio of length to breadth beyond which a plate may be considered as a long narrow plate for the purpose of calculating deflections and stresses in its central portion. The formulas of these last two sections are to be considered as only moderately accurate approximations, permitting an estimation of the relation between deflection and load.

Assumptions Made Regarding Properties and Structure of Wood.

In the analysis wood is taken to be an orthotropic material, i.e., a material having three mutually perpendicular planes of elastic symmetry. The effect of the glue other than that of securing adherence of adjacent plies is assumed to be negligible. Consequently, the formulas and methods of this report are not intended to apply directly to partially or completely impregnated plywood or compregnated wood, although it is to be expected that many of the results can be applied to such material. Under these assumptions the differential equations are set up for the determination of the deflection of plywood plates for the cases of both small and large deflections.

The form of the differential equation for the deflection of an orthotropic plate, in the case of small deflections, is well known. For plywood plates, which are made up of layers of orthotropic material, the coefficients in the differential equation are given in terms of the elastic constants of the constituent wood in the author's paper, "Bending of a Centrally Loaded Rectangular Strip of Plywood." The derivation of these coefficients under certain simplifying assumptions is given in the present report. Although actual plywood will seldom possess the structure assumed as ideal, nevertheless the procedure used in arriving at the coefficients brings to light the essential factors determining the stiffness and other elastic properties of plywood. Then, in a given situation a rational allowance for the effects of variation from the ideal structure can be made.

The principal results are given in the body of the report while the mathematical analysis leading to them is placed in a series of appendixes whose numbers are the same as those of the corresponding sections of the text.


2 Physics 7, 32-41, 1936.
In the numerical calculations, made to illustrate the application of the formulas, it is necessary to use a species of wood for which the appropriate elastic constants are known. For plywood with flat-grain plies, five of the twelve elastic constants of the wood of the species under consideration are needed, namely, two Young's moduli, two Poisson's ratios, and one modulus of rigidity. The elastic constants of spruce have been determined carefully. It is for this reason that the illustrative calculations have been made for plywood plates of spruce.

In a number of instances the numerical results are expressed in a form involving a factor containing Young's modulus along the grain. It is to be expected that satisfactory values of the corresponding results can frequently be obtained for plywood made from wood of another species by replacing the Young's modulus along the grain for spruce by the corresponding modulus for the second species. When this is done it must be realized that the assumption is made that the elastic moduli all change in the same ratio in passing from one species to the other and that the relevant Poisson's ratios are the same for the two species. Experience will show that considerable variations from this assumed relationship can occur without greatly affecting certain of the results. However, the basic formulas take into account the elastic properties of the particular species under consideration. The behavior of plates made up of plies of wood of two different species can also be determined with the aid of these formulas.

Section 1. The Elastic Behavior of Wood

The visible structure of wood suggests that it may be considered to have three mutually perpendicular planes of elastic symmetry, namely, the planes perpendicular to the longitudinal, radial, and tangential directions, respectively, as shown in figure 3. A substance having such properties of elastic symmetry is said to be orthotropic.

If wood is orthotropic it will have (see appendix 1) three Young's moduli, \( E_L, E_R, \) and \( E_T \), the letters \( L, R, \) and \( T \) denoting the longitudinal, radial, and tangential directions, respectively; three shearing moduli \( \mu_{LT}, \mu_{LR}, \) and \( \mu_{RT} \); and six Poisson's ratios, \( \sigma_{LT}, \sigma_{TL}, \sigma_{RT}, \sigma_{TR}, \sigma_{LR}, \) and \( \sigma_{RL} \) where, for example, \( \sigma_{LT} \) is the Poisson's ratio associated with tension parallel to the direction \( L \) and contraction parallel to the direction \( T \). Among these 12 constants there are three relations of the type (see (1.7) in appendix 1).

\[
E_L \sigma_{TL} = E_T \sigma_{LT}.
\]

Table 1, giving the values of these constants for several species of wood as determined on the assumption that wood is orthotropic, is taken from a report by C. F. Jenkin "Report on Materials of Construction Used in Aircraft," Aeronautics Research Committee (London, 1920).²

²See also H. Carrington, Phil. Mag. 41, 206, 348, 1921; 43, 371, 1922; 44, 288, 1922; 45, 105, 1923.
The values given below of the elastic moduli of Douglas-fir at 10 percent moisture content, were obtained from a limited number of tests. Because of the small number of tests, these values are to be considered as tentative.

\[ \text{Lb./in}^2 \]

\[ E_L = 1,960,000 \]
\[ E_T = 113,200 \]
\[ E_R = 155,800 \]
\[ \mu_{LT} = 123,800 \]
\[ \mu_{LR} = 110,600 \]
\[ \mu_{RT} = 7,100 \]

The Poisson's ratios \( \sigma_{LT} \) and \( \sigma_{TL} \) have not been determined. In the calculations made later in this report they are taken to be the same as for spruce. It is probable that the error thus introduced is not large.

The assumption that wood may be treated as an orthotropic material is reasonably well confirmed by the experimental evidence at present available.

**Plates with Small Deflections**

**Section 2. The Differential Equation for the Deflection of a Plywood Plate.**

**Small Deflections**

In deriving the differential equation for the deflection of a plywood plate the usual assumptions underlying the theory of thin plates are made. In addition, wood is taken to be an orthotropic substance and the following assumptions are made concerning the structure of the plywood:

The material of the individual plies is accurately flat-grain, that is, the directions of the grain and of the annual rings are parallel to the faces of the plies. The directions of the grain in adjacent plies are perpendicular to each other and parallel or perpendicular to the respective edges of the plate. The analysis applies equally well to edge grain plywood. It is only necessary to substitute \( R \) for \( T \) throughout in the subscripts of the elastic constants.

Each ply is homogeneous. This implies that the variations of the elastic constants from springwood to summerwood are disregarded and average values of the constants are used.

---

The plate is symmetrical, both geometrically and as to arrangement and properties of the material, with respect to the plane \( z = 0 \), the axes of coordinates being chosen as in figure 1. If a plate is not of symmetrical construction with respect to the plane \( z = 0 \) approximately correct results should be obtained by using in place of the flexural rigidities \( D_1 \) and \( D_2 \) the flexural rigidities of strips of unit width parallel to the edges of the plate.

The elastic constants of the wood are the same in all plies. This assumption can be omitted without materially complicating the discussion, provided that the other assumptions are retained.

Let \( h \) denote the thickness of the plate and \( p \) the load per unit area acting normal to the face \( z = -h/2 \) in the direction of the positive axis of \( z \) in figure 1. The deflection \( w \) of points in the middle surface satisfies the differential equation

\[
D_1 \frac{\partial^4 w}{\partial x^4} + 2K \frac{\partial^2 w}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 w}{\partial y^4} = p,
\]

which is derived in appendix 2. The flexural rigidities \( D_1 \) and \( D_2 \) are proportional to two "mean moduli" in bending" \( E_1 \) and \( E_2 \) as explained in appendix 2 where the plate is assumed to be of symmetrical construction with respect to the middle plane, \( z = 0 \).

Equation (2.12) may be expected to apply with small error to plates of unsymmetrical construction if \( E_1 \) and \( E_2 \) are determined from the flexural rigidities of strips of unit width parallel to the \( X \) and \( Y \) axes, respectively.

Thus, for a strip of unit width parallel to the \( X \)-axis, \( E_1 \) is defined by the equation

\[
E_1 I = \frac{2}{7} (E_x) \frac{I_1}{I_1}
\]

An equation of this form for orthotropic material is well known. See, for example, the references to Houbert on page 4. The dependence of the coefficients in this equation upon the elastic constants of wood and upon the structure of the plywood plate is discussed in appendix 2.

Price, A. T., Phil. Trans. A. 228, 1, 1928. The definition here given differs somewhat from that used by Price for the apparent Young's modulus in bending. The essential difference is in a term whose value is small. See his equations (13.82) and his discussion of plywood on pages 50 and 52.
where the summation is extended over all of the plies; \((E_x)_i\) is the Young's modulus of the \(i\)th ply measured parallel to the \(X\)-axis; \(I_i\) is the moment of inertia with respect to the neutral axis, of the cross section of the \(i\)th ply made by a plane perpendicular to the \(X\)-axis; and \(I = h^2/12\) is the moment of inertia of the entire cross section with respect to the central line \(z = 0\). An approximate formula in which the error is very slight is obtained for \(E_1I\) by taking the sum of the products \((E_x)_iI_i\) formed for only those plies in which the grain is parallel to the length of the strip. Exception is to be made of a three ply strip having the grain of the face plies perpendicular to the length of the strip. The flexural rigidity \(E_2I\) is calculated in a similar way.

These definitions for \(E_1\) and \(E_2\) may also be applied in dealing with plates of symmetrical construction. In the case of such plates they are identical with the definitions of equations (2.18) and (2.19) of Appendix 2.

In order to have a definite situation in mind it will be assumed from this point on, unless the contrary is explicitly stated, that the plates are of symmetrical construction and that all plies are of the same thickness. However, the application of the formulas obtained is not limited to plates of this assumed structure.

Irregularities in the state of stress at the junction of two plies were neglected in deriving equation (2.12). This situation is discussed on pages 14, 15, 50, and 51 of the paper by Price, to which reference is made in footnote 6. The effect of these irregularities in the state of stress would be to increase slightly the flexural rigidities of the plate above those calculated from the mean moduli \(E_1\) and \(E_2\). These effects could not be clearly detected in a long series of static bending tests of strips of plywood and are, therefore, considered to be so small that they may be neglected.

With the elastic constants for spruce and for Douglas-fir as previously given the values for \(E_1\) and \(E_2\) in tables 2 and 3 were found, the grain of the face plies being parallel to the \(X\)-axis.

The differential equation (2.12) can be reduced to the simpler form

\[
\frac{\delta^4 W}{\delta x^4} + 2\kappa \frac{\delta^4 W}{\delta x^2 \delta\eta^2} + \frac{\delta^4 W}{\delta\eta^4} = \frac{P}{D_1} \tag{2.25}
\]

where

\[
\kappa = K / (D_1 D_2)^{1/2} \tag{2.26}
\]

by making the substitution

\[
\eta = \varepsilon y \tag{2.27}
\]
where

\[
\varepsilon = \left( \frac{D_1}{D_2} \right)^{1/4} = \left( \frac{E_1}{E_2} \right)^{1/4}
\]

The equation (2.25) (or 2.12) is to be solved under appropriate boundary conditions to determine the behavior of a given plate under a given load.

In deriving this differential equation it has been assumed that the deflections are so small that direct stresses are not developed to an appreciable extent. This implies maximum deflections of, roughly speaking, less than one-half of the thickness of the plate.

After equation (2.25) has been solved for the deflection as a function of the coordinates, the components of stress can be found with the aid of equations (2.5). Or the stresses can be expressed in terms of the bending and twisting moments which in turn are expressed in terms of the deflection in equation (2.23).

Section 3. Rectangular Plate Under Uniformly Distributed Load. Edges Simply Supported

In this case the edges of the plate \( x = 0, x = a, y = c, \) and \( y = b, \) are simply supported and the load \( p \) per unit area is a constant, the same at all points of the plate. The deflection is found as the solution of the differential equation (2.25) subject to the conditions that on the edges \( x = 0 \) and \( x = a, \) the deflection \( w \) and the bending moment \( m_x \) vanish and that on the edges \( y = c \) and \( y = b, \) the deflection \( w \) and the bending moment \( m_y \) vanish. The plate is assumed to be held down at the corners.

The exact solution (3.15) mentioned briefly in the earlier part of this section and discussed in greater detail in appendix 3 is due to D. E. Zilmer who, as a graduate student at the University of Wisconsin on a fellowship supported by the Forest Products Laboratory, undertook the solution of this problem and that of the clamped plywood plate (see section 4) at my suggestion. The calculations based on this exact solution were performed by the Computing Division of the Laboratory. The present author is responsible for the approximate method and for the discussion of this section and that of appendix 3. Solutions of the problem for the simply supported plate of orthotropic material have also been given by Huber and by Iguchi in the papers to which reference was made in footnote 1.

The reader who wishes a quick and easily applied method of finding the approximate deflection at the center of a plate should turn at once to page 13.
Since the differential equation is written with the variable $\eta$ instead of the variable $y$ it is convenient to think of the plate as having been transformed into one having as edges the lines $x = 0$, $x = a$, $\eta = 0$, and $\eta = \beta$ where

$$\beta = \varepsilon b$$

(3.1)

and to express the deflections and moments in terms of the variables $x$ and $\eta$ instead of in terms of the variables $x$ and $y$. The deflection is found to be given by the equation (see appendix 3).

$$w = \frac{48\varepsilon}{\pi^2} \frac{E}{h^2} P \frac{1}{n^3} \lambda_n \sin \lambda_n x$$

(3.15)

$$n = 1, 3, 5 \ldots$$

where

$$P = \frac{p a^4}{E h^4}, \quad \lambda = 1 - \frac{\sigma h}{E T}$$

(3.16)

$$\lambda_n = n\pi/a$$

(3.7)

and $Y_n$ is defined by (3.12) appendix 3.

The bending moments $m_x$ and $m_y$ and the twisting moment $m_{xy}$ can be calculated from (3.15) with the aid of (2.29) of appendix 2.

The ratio of the deflection $w_0$ at the center, to the thickness is readily calculated from (3.15) and found to be given by equation (3.15) in appendix 3 in terms of an infinite series which converges so rapidly that it may usually be replaced by its first term only.

Approximate formula for the deflection at the center.—As in many cases the satisfactory behavior of a plate will be determined by its deflection at the center, it is desirable to have a simple approximate formula for calculating this deflection. Such a formula can be obtained by making use of the fact that in a configuration of equilibrium the sum of the potential energy of deformation of the plate and that of the applied load is a minimum as compared with other configurations satisfying the same boundary conditions. In the application of this principle a plausible simple form is assumed for the deflected middle surface of the plate, this form containing the deflection $w_0$ at the center as an undetermined parameter. The potential energy of deformation of the plate in bending $V_b$ and the potential energy of the load $V_l$ are then calculated as a function of $w_0$. The total potential energy $V$ of the system is the sum of $V_b$ and $V_l$. On equating to zero the derivative of $V$ with respect to $w_0$ an equation is obtained connecting $w_0$ with $p$, the applied load per unit area. A better approximation to the form of the deflected middle surface is obtained by assuming for it an expression containing several parameters.
and determining all these parameters in such a way that the total potential energy is a minimum. In the following analysis a second parameter $\tau$ (or $c$) whose significance will be explained is introduced in addition to the parameter $w_0$, the deflection at the center.

Let $$w = w_0 \sin \frac{\pi x}{a}$$

when $$a < y < b - a$$

$$w = w_0 \sin \frac{\pi x}{a} \sin \frac{\pi y}{2c}$$

when $$0 < y < c.$$ (See figs. 4 and 5.) An expression corresponding to the latter is assumed for the portion of the plate for which $b - c < y < b$. This need not be written down since the potential energy of the whole plate can be calculated as twice the potential energy of the half of the plate for which $y < b/2$. It is convenient to introduce the parameter $\tau$

which is connected with $c$ by the equation $c = \frac{\tau a}{2}$. Thus the parameters $\tau$ and $w_c$ are to be determined in such a way that the total potential energy of the system is a minimum. The form of the assumed middle surface along the line $x = a/2$ is shown in figure 5.

In the form assumed for the deflected middle surface, the curvature is discontinuous along the lines $y = c$ and $y = b - c$. This discontinuity could be removed by a slight change in the form of the deformed middle surface in the vicinity of these lines with a consequent small change in the total potential energy.

After making the necessary calculations (see appendix 3) the following formula is found connecting $w_0/h$ and $p$.

$$w_0/h = \alpha P$$  \hspace{1cm} (3.37)

where

$$P = \frac{P_0 a^2}{E_L L^4}$$  \hspace{1cm} (3.38)

where $\alpha$ is given by (3.36) in appendix 3.

Tables 4 - 9 were calculated using the values of the elastic constants of spruce. Values of the factor $\alpha$ as computed from the exact and approximate formulas are also given in these tables. It appears that the approximate formulas are reasonably accurate. The factor $\alpha$ is obtained from the exact formula (3.15) as $(w_0/h)/P$.

**Form of the Deflected Surface**

Examination of the exact values of the factor $\alpha$ for the plywood plates discloses an interesting and unexpected phenomenon. The exact value
of $\alpha$ and hence of the central deflection $w_0$ for a given load increases to a maximum value as $k = b/a$ increases to a particular value and then decreases asymptotically to a limiting value as $k$ is further increased. If the calculations for plate 3Y had been carried out for larger values of the ratio $k = b/a$ the phenomenon would have been found in this case also, as rough calculations show.

The possibility of such a behavior is revealed by a careful examination of formula (3.15) for the deflection. The term $Y_n$, given in equation (3.12) of appendix 3, contains trigonometric functions of the variable $\eta$. This implies the possibility of a wave form along any line $x = constant$ and in particular along the central line $x = \frac{a}{2}$. Because of the presence of this wave form the deflection at the center does not increase steadily to an asymptotic value with increasing ratio of length of plate to breadth. For a certain value of this ratio the central deflection has a maximum which is greater than the deflection at the center of a very long plate, under the same uniform load per unit area. The trigonometric terms are not present in the term $Y_n$ of the formula to which (3.15) reduces for the isotropic plate. Hence a wave form of the deflected surface along the line $x = constant$ is not to be expected in this case.

A wave form in the surface of an orthotropic plate under concentrated load was noted by Huber. At my suggestion D. E. Zilmer investigated carefully the behavior of uniformly loaded plates of the type 3X. He found that terms after the first in equation (3.15) could be neglected in studying an effect of the order of magnitude under consideration so that the wave form indicated by the first term of this equation could not possibly be obliterated by subsequent terms. He found that the central deflection considered as a function of $k = \frac{b}{a}$, attained a maximum value at $k = 1.49$ that was about 3 percent greater than the asymptotic value of this deflection for large $k$. This conclusion agrees with the results of table 5 for plate 3X. Table 7 for plate 5X shows that the maximum central deflection as a function of $k = b/a$ occurs for $k = 2$ approximately and that this maximum deflection is 5.7 percent greater than the asymptotic value of the central deflection for large $k$.

In a number of tests with plates of commercial plywood the surfaces were observed to take wave forms. The material of the plates was not sufficiently uniform to warrant comparison of the observed wave form with that predicted by the formula, since the effect predicted is so small that it would be easily masked by small variations in the material of the plates.

In figures 6 to 9 are shown the deflection along the central line $x = \frac{a}{2}$ of a number of plates of commercial plywood under uniformly distributed load with differing ratios of length to breadth. In order to compare the shapes of these sections of the deformed surfaces, dial readings corresponding to the same central deflection are plotted and the distances from one end of the plates have been expressed as fractions of $b$, the length of the plate. The method of loading the plates and measuring the deflections is described in the latter part of this section.

g. Thesis to be presented at the University of Wisconsin.
The curves for plates Nos. 4 (type 3X), 6 (type 5X), and 8 (type 5X) show clear indications of a change in shape of the deflected surface associated with the presence of a wave form. This effect does not appear clearly in the curves of figure 7 for plate No. 5 (type 5Y). According to table 6 the maximum effect of wave form may be expected for larger values of $k = b/a$ in the case of plates of type 5Y than in the case of those of type 5X.

Central Deflection of a Plate Expressed as a Fraction of That of the Corresponding Infinite Strip. Determination of $w_0/h$ from a Curve.

The deflection $w_o$ at the center of a rectangular plate, simply supported at its edges and under a uniformly distributed load, can be expressed as a fraction of the deflection along the central line of an infinitely long plate similarly loaded. From this standpoint the deflection of a finite plate is regarded as that of an infinite strip multiplied by a corrective factor to take account of the effect of the ends of the finite plate. The deflection at the center of a uniformly loaded infinite strip simply supported at its edges is given by the formula

$$w_o = \frac{5(1 - \sigma \frac{L^4}{12}) \frac{p}{E_1 h}}{32} = 0.1547 \frac{p a}{E_1 h^3}$$ (3.39)

This is the formula for the central deflection of a uniformly loaded beam of unit width except for the factor in parentheses. This factor has been taken to be 0.99, the value which it has for spruce.

If the deflection at the center of a finite rectangular plate is denoted by $w_o$ we can write

$$w_o = \gamma w_o \infty$$ (3.90)

The factor $\gamma$ is found to depend almost entirely upon the value of $\beta/a = (b/a)(E_1/E_2)^{1/4}$, the ratio of the sides of the transformed plate and very little upon the type of plywood in the plate except insofar as this influences the value of the ratio $\beta/a$.

The curve of figure 10, representing $\gamma$ as a function of $\beta/a$ is a smooth average curve for points determined from the exact values of $\alpha$ in tables 4 - 9. These points are shown in the figure. More points than those obtained directly from the tables were secured by interchanging the

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10 A presentation of the results of an approximate analysis in essentially the form (3.90) was made by Norris, C. B., Hardwood Record, May 1937. Because of the approximations involved, the deflections calculated from his results are too small, a fact which he recognized would be the case.
axes to which the plates in the tables were referred and utilizing the data of the tables to calculate the factor $\gamma$ for plates for which the ratios $b/a$ of the actual dimensions were less than one.

The factor $\gamma$ was also calculated from the results of the approximate formulas. This was done for plates $3X$, $3Y$, $5X$, $5Y$, and $7X$ with all plies of the same thickness and also for plates $3X$, $5X$, and $7X$ with the face plies one-half as thick as the remaining plies. All of these points are shown in figure 11. The curve in this figure is that of figure 10, namely, the average curve for points determined by the exact formulas.

The curve is evidently sufficiently accurate for types of plywood similar to those under consideration in this report. The controlling elastic properties of the plate are manifested in the stiffness in the direction parallel to the $X$-axis which determines the deflection of the infinite strip, and in the ratio of the stiffnesses in the $X$ and $Y$ directions as it appears in the factor $\varepsilon = (E_1/E_2)^{1/4}$ which is used to obtain $\beta = \varepsilon b$ of the transformed plate. Hence, to use the curve of figure 10 for a given plate it is only necessary to know the two nonidentical Young's moduli in bending, $E_1$ and $E_2$, of the plate. They can be determined from static bending tests or estimated from the structure of the plywood in the plate. The factor $\gamma$ corresponding to $\beta/a = b/a (E_1/E_2)^{1/4}$ can then be read from the curve. The central deflection $w_0$ is then given by

$$w_0 = \gamma w_0 = \infty$$

where $w_0 = \infty$ is to be calculated by equation (3.39).

The curve of figure 10 can be constructed from the values in table 10.

Tests

In table 11 are shown the results of a number of tests made with uniformly loaded plates of commercial plywood. A description of the method of making the tests will be given below. The factor $\gamma_{obs}$ was calculated as follows. The tests on a given plate yielded a mean value for the ratio $p/w_0$ where $p$ is the load per unit area and $w_0$ is the deflection at the center. The moduli $E_1$ and $E_2$ were determined by static bending tests on strips cut from the plates after the tests on the plates themselves were completed. From the formula

$$w_0 = \frac{5(1 - \sigma_L T \sigma_{TL})}{32 \frac{pa^4}{E_1 h^2}}$$

The tests described here and in sections 4 and 7 were carried out under the direct supervision of Alan D. Freas, Assistant Engineer.
we obtain

\[ (\gamma)_{\text{obs}} = \frac{w_0}{w_0^{\infty}} = \frac{32E_1 h^3}{5(1 - \nu_{LT^T})a^4} \frac{w_0}{p} \]

The values of \( \gamma \) thus found were compared with those of \( \gamma_{\text{theor}} \) obtained from the curve for \( \gamma \) as a function of the ratio \( \beta/a \). As we have seen this curve represents a fairly good approximation to the theoretical values of \( \gamma \). In all instances the ratio of \( \gamma_{\text{obs}} \) to \( \gamma_{\text{theor}} \) is less than unity. The fact that the observed deflections are smaller than they would be expected to be can be attributed to a certain amount of restraint at the edges which could not be entirely eliminated. The variability of the results can be attributed partly to lack of uniformity of the plywood in a given plate and partly to varying degrees of constraint at the edges of the plate.

It appears that the curve for \( \gamma = w_0/w_0^{\infty} \) may be used in predicting the deflection at the center of a plate if reasonable allowance is made for the effect of constraints at the edges and for variability of the material.

In making the tests the plates were placed between two rectangular frames made of heavy channels. The frames were 12 feet long and 4 feet wide. A cross section of the apparatus is shown in figure 12. For the case of simply supported edges the plate rested on circular rods 1/2 inch in diameter. The pressure was applied by inflating three rubber bags, approximately 4 feet square and 6 inches deep, with compressed air. Heavy planks bolted to the channels as shown in the figure formed the back of the chamber containing the bags. A run was made with the load on one side of the plate and then on the other side by moving the planks and bags. The plates were tested in the vertical position to eliminate the effect of gravity. The deflections were read on Ames' dials placed at various positions on the plates. The air pressure was measured with a water manometer. Tests were first made on a 12 by 4-foot plate. Then 4 feet were sawed off and the resulting 8 by 4-foot plate tested. Finally this plate was sawed in two and tests were run on a 4 by 4-foot plate. As the 12 by 4-foot plates were made by joining up shorter lengths of plate by scarf joints there were frequently considerable variations in the elastic constants from one end of the plate to the other. In addition, there were present defects in manufacture and variations in direction of grain.

Section 4. Rectangular Plate Under Uniform Load. Edges Clamped

In this case the deflection \( w \) and its normal derivatives vanish along the edges of the plate. The solution of the differential equation (2.25) subject to these boundary conditions will be found in appendix 4.

\[^{12}\text{This solution is due to D. E. Zilmer. (See footnote 7, p. 9.) The calculations based on this exact solution were performed at the Forest Products Laboratory under the direction of the present author who is responsible for the discussion to be found in section 4 and in appendix 4. He is also responsible for the approximate methods. The clamped orthotropic plate was also treated by S. Iguchi by a somewhat different method in the paper to which reference was made in footnote 1. See also footnote 28.}\]
Approximate formula for the deflection at the center. As in the case of the plate with simply supported edges it is possible from a consideration of the potential energy of the system to find a simple approximate formula for the deflection at the center of a uniformly loaded plate with clamped edges. The procedure is the same as that employed in section 3. The plate is divided into three regions by the lines \( y = c \) and \( y = b-c \) where \( c = \tau a/2 \), \( \tau \) being a parameter to be determined. The assumed form of the deflection of the middle surface is given by the equations

\[
\eta = \eta_0 \sin^2 \frac{\pi x}{a}
\]

when \( 0 < y < b \) \( c < y < b-c \) \( \text{(4.22)} \)

\[
\eta = \eta_0 \sin^2 \frac{\pi x}{2c} \sin^2 \frac{\pi x}{a}
\]

when \( 0 < y < c \)

An expression corresponding to the latter is assumed for the region \( b-c < y < b \).

After making the necessary calculations (see appendix 4) the following formula is found connecting \( \eta_0/h \) and \( \rho \).

\[
\frac{\eta_0}{h} = \alpha \rho \quad \text{(4.33)}
\]

where

\[
\rho = \frac{4a^2}{E_\perp h^4}
\]

and \( \alpha \) is given by formula (4.32) of appendix 4. Values of the factor \( \alpha \) as computed from the exact and approximate formulas are given in tables 12-20. The exact factor \( \alpha \) is obtained from the exact formula (4.11) together with (4.4) and (4.6) as \( (\eta_0/h)/\rho \). It appears that the approximate formulas are reasonably accurate. The tables were calculated using the elastic constants of spruce. The number \( k \) denotes the ratio \( b/a \) of the sides.

Form of Deflected Surface

Examination of the exact values of the factor \( \alpha \) for plates of type 3X and 5X shows that here, just as in the case of plywood plates with simply supported edges, the deflection for a given load increases to a maximum value as \( k = b/a \) increases to a particular value and then decreases to an asymptotic value as \( k \) is further increased. This effect

\( ^{13} \) An approximate method for finding the deflection at the center, which requires no knowledge of the mathematical analysis, will be found on page 17.

-16-
is somewhat more pronounced than in the case of plates with simply supported edges. As before, this effect is to be associated with the fact that the deflected surface assumes a wave form. The possibility of the existence of such a wave form is shown by the presence of trigonometric functions in the expression for the deflection.

In figures 13, 14, and 15 are shown the deflection along the central line \( x = a/2 \) of a number of plates of commercial plywood. In order to compare the shapes of these sections of the deformed surfaces dial readings corresponding to the same central deflection are plotted. The distances from one end of the plates have been expressed as fractions of \( b \) the length of the plate.

The curves show even more pronounced indications of a change in shape of the deflected surface associated with the presence of a wave form than those for plates with simply supported edges. In fact, the effect is so pronounced in the case of plate \( \frac{1}{4} \) that one hesitates to accept it as real. That this effect, which is so pronounced in the case of the \( \frac{1}{4} \) by 8-foot plate, is actually present in this plate at all stages of loading is shown in figures 16 and 17, which give the shape of the plate at successive intervals of loading. In order to compare readily the shapes of the curves the central deflections have been reduced to 0.100. This means that for small deflections experimental errors have been multiplied by a large factor. The curves for this plate are published without further comment merely to show what actually happened in the case of this particular plate. It may be remarked that for the larger deflections the relative heights of the maxima are reduced. This may presumably be attributed to the effect of membrane stresses.

Central Deflection of a Plate Expressed as a Fraction of That of the Corresponding Infinite Strip.

As in the case of the plate with simply supported edges it is possible to represent the deflection at the center of a plate with clamped edges as a fraction \( \gamma \) of the deflection at the center of an infinite strip with clamped edges. Thus

\[
\frac{w_0}{w_0} = \gamma
\]

(4.35)

The factor \( \gamma \) is again found to depend almost entirely upon the ratio \( \beta/a = (b/a)(E_1/E_2)^{1/4} \) so that the results of the theory, so far as deflection at the center of the plate is concerned, can be represented with sufficient approximation by a curve in which \( \gamma \) is plotted as a function of \( \beta/a \). This curve, constructed from the exact values of \( \alpha \) in tables 12 to 15, is shown in figure 18. More points than those obtained directly from the tables were secured by interchanging the axes to which the plates in the tables were referred and utilizing the data of the tables to calculate the factor \( \gamma \) for other values of the ratio \( \beta/a \).
For an infinite strip clamped at the edges, the deflection at a point on the central line is given by the formula

\[ w_{00} = \frac{1}{32} \frac{1 - \nu_{LT}^2}{E_1 h^3} TLa^4 \]  

(4.36)

In which the factor \( 1 - \nu_{LT}^2 \) may be taken to be 0.39. Except for this factor formula (4.36) is that for a uniformly loaded beam of unit width with fixed ends.

The curve of figure 18 can be constructed from the values in table 21.

In figure 19 the curve is that of figure 18 and the points are those computed from the values of \( \gamma \) given by the approximate formula, the values of \( w_{00} \) being those given by the exact formula. Except for the fact that the approximate values of \( \gamma \) do not show a maximum in the vicinity of \( \beta/a = 2 \) the agreement is satisfactory. That the approximate values of \( \gamma \) do not show a maximum in the vicinity of \( \beta/a = 2 \) is to be attributed to the incomplete representation of the deflected surface by the forms assumed in (4.22). Since the exact analysis clearly points to the existence of a maximum point on the curve, it may safely be assumed that the curve represents approximately the true situation for the seven-ply and nine-ply plates in addition to that for the three-ply and five-ply plates for which it was constructed.

Tests

In table 21 are shown the results of a number of tests of uniformly loaded plates with clamped edges. The clamping at the edges of the plates was accomplished by removing the circular rods shown in figure 12 and clamping the plate between the channels of the two frames. The same plates of commercial plywood were used in these tests as in the tests of plates with simply supported edges.

The factor \( \gamma_{obs} \) was computed from the mean of the values of the quantity \( p/w_0 \) for a given plate by the formula

\[ \gamma_{obs} = \frac{w_0}{w_{00}} = \frac{32E_1 h^3}{(1 - \nu_{LT}^2)a^4} \frac{w_0}{p} \]

The corresponding factor \( \gamma_{theor} \) was taken from the curve of figure 18 for the appropriate value of \( \beta/a \). As was to have been expected, owing to the impossibility of securing perfect clamping at the edges, the
observed factors $\gamma$ and consequently the observed central deflections are greater than those predicted by the curve, which is based on the exact analysis of ideal cases. On the average they are about 40 percent greater.

Hence in using the results for plates with clamped edges considerable allowance must be made for the effect of imperfect clamping. The formula predicts the central deflection for the case of perfect clamping, a situation that is rarely met in practice. It is realized in the case of a plate extending over a network of rectangular openings, all in the same plane. Ideal clamping will be found on the edges of interior rectangles of such a network. Otherwise elastic yielding reduces the clamping effect to a greater or less extent, depending upon the particular situation at the edges in a given case.

Section 5. Infinite Strip (Long, Narrow, Rectangular Plate). Load Concentrated at a Point. Edges Simply Supported

Consider an infinite plywood strip with edges $x = 0$ and $x = a$ along which it is simply supported and under a concentrated load $P$ applied at the point $x = u$, $y = 0$ on the $X$-axis as in figure 30. As in the case of the isotropic strip, a solution of the differential equation (2.25) is obtained for the case in which a load of uniform intensity is distributed along a segment of the $X$-axis including the point $(u, 0)$ in its interior. By allowing the length of the segment to decrease while at the same time the intensity of the load increases in such a way that the total applied load is unchanged, we obtain the solution for the limiting case of a point load. This solution, expressed in terms of an infinite series, is given by (5.7) of appendix 5.

From this expression for the deflection, the bending moments $m_x$ and $m_y$ can be calculated. It is found that it is possible to express the sums of the infinite series for these moments in closed forms in terms of two functions which are the real and imaginary components of a function of a complex variable. Replacing the series by closed forms reduces greatly the necessary calculations. (See appendix 5.) It is clear that these moments should become infinite at the point of loading as they do.

The values of the deflection and bending moments at certain points of an infinite strip of plywood of type 3X having a concentrated load at a point on its central line are given in table 23 and shown in the curves of figure 20.


A uniform load acts over a small rectangular portion, as shown in figure 21, of an infinitely long strip whose edges are simply supported. By integrating the effect of the loading of a narrow strip of the rectangle, considered as a loaded line segment, formulas are obtained in

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Nadai, A., Elastische Platten, pp. 78-82 and 85-95.
appendix 6 for calculating the resulting deflections and moments at any point of the infinite plate. From these formulas the moments can be calculated with the aid of equations (2.29). Table 24 and the curves of figure 22 give the deflections and moments at certain points due to a load distributed uniformly over a small square whose center is on the center line of a three-ply plate of spruce plywood and whose sides are equal to one-tenth of the width of the plate. If table 24 and figure 22 are compared with table 23 and figure 20, the deflections in the case of a load applied over a small square area are seen to be practically identical with those due to a similarly situated point load equal to the total load applied over the square. The bending moments are also practically the same except in the immediate vicinity of the loads.

Section 7. Rectangular Plate. Load Concentrated at a Point or Applied Over a Small Area. Edges Simply Supported

The deflections and moments due to a concentrated load on a rectangular plate with simply supported edges can be found by calculating the effects of a suitable distribution of positive and negative loads on an infinite plate, using the results of section 5 or section 6. It is only necessary to distribute the loads in such a way that the deflections and bending moments vanish on the edges, \( y = 0 \) and \( y = b \) of the plate. The distribution is shown in figure 23. A positive load is denoted by a dot (\( \cdot \)) and a negative load by a cross (\( \times \)). If the loads are numbered I, II, III, IV, V, etc., as shown in the figure, the deflection \( w \) at any point in the plate will be given by combining the deflections due to the separate loads, that is,

\[
  w = w_I + w_{II} + w_{III} + \cdots
\]

In like manner expressions for the bending moments can be obtained.

Calculations were made for a square plate of type 3X having in one case a point load at its center and in another case a uniform load distributed over a central square whose sides were taken to be one-tenth of those of the plate. The results are shown in tables 25 and 26 and in the curves of figures 24 and 25. The choice of axes is to be noted as slightly different from the customary choice for a finite plate in this report.

The calculations for the case of a uniformly loaded small central square area have not been carefully checked. However, the behavior indicated by the results is in good agreement with what was to be expected from the other cases considered in this section and in section 6.

In the neighborhood of the load the deflections for the square plate are nearly the same as for the infinite plate while near the edges the effects of the loads V and II must be taken into account. In case of a five-ply square plate the effect of these loads would be noticed at greater distances from the edges of the plate.
Central Deflection of a Plate Expressed as a Fraction of That of the Corresponding Infinite Strip

The maximum deflection of a rectangular plate under a given concentrated load \( P \) occurs at the center of the plate with the concentrated load placed at the center. For an infinite strip the central deflection due to a point load on the central line is given by (see (7.4) appendix 7)

\[
w_{o\infty} = 1.051 \frac{Pn^2}{E_1h^3} \frac{6\lambda_\alpha a^2}{\rho n^2}
\]  

(7.4)

For a finite rectangular plate under a given central load concentrated at a point, the central deflection \( w_o \) can be expressed as a fraction \( \gamma \) of the central deflection of a similarly loaded infinite strip. Thus

\[
\gamma_w = \gamma w_{o\infty}
\]

(7.5)

For the purpose of determining this factor \( \gamma \), it is advantageous to replace the method explained earlier in this section by one used by Timoshenko\(^{15}\) for isotropic plates. In this method the load is limited to a position on the central line. In appendix 7 the analysis is carried out for a plywood plate. The calculation of the factor \( \gamma \) in (7.5) can then be readily performed. (See appendix 7.) It is found that \( \gamma \) can be represented with little error by a curve as a function of the ratio

\[
\beta/a = (b/a)(\frac{E_1}{E_2})^{1/4}
\]

In table 27 are given the values of \( \gamma \) for plates of several types of plywood. In figure 26 a smooth average curve for \( \gamma \) is drawn from the points given in table 27. Further calculations indicate that this curve is satisfactory for other types of plywood than those listed in table 28. This curve should be used only for plates of the types considered in this report. In particular, the directions of the grain of the wood in adjacent plies are mutually perpendicular and the constant \( \lambda \) lies between 0.2 and 0.5 or not far outside of this interval. The results of tests on a number of plywood plates with concentrated central loads are shown in table 28. The same plates of commercial plywood were used as in the tests described in sections 3 and 4. The plates were tested in a horizontal position. The edges were simply supported, resting on half-inch rods as shown in figure 12. The loaded area was in all cases a square \( 4 \) by \( 4 \)-inches. Only one side of each plate was loaded. The numbers in the column headed \( \gamma_{obs} \) were calculated from the observed ratios \( w_o/P \) by the following formula which is obtained from (7.4):

\[
\gamma = \frac{w_o}{w_{o\infty}} = \frac{3E_1h^3}{(1.051)6\lambda_\alpha a^2} \frac{w_o}{P}.
\]

\(^{15}\) Timoshenko, S., Bauingenieur, 3, 51, 1922.
The numbers of the column headed $\gamma_{\text{theor}}$ were taken from the curve of figure 26. The agreement between the observed and theoretical values of $\gamma$ appears to be satisfactory. The average value 0.921 of the ratio $\gamma_{\text{obs}}/\gamma_{\text{theor}}$ indicates some restraint at the edges but not so much as in the case of plates with uniformly distributed loads. This appears to be reasonable.

Plates With Large Deflections

Section 8. Differential Equations for the Deflection of a Plywood Plate. Large Deflections

When a plate with prescribed edge conditions is subjected to a succession of increasing loads it is known that at first the deflection at the center of the plate increases proportionally to the load but that it does so only during the early stages of the loading. As the load is increased, the stresses remaining below the proportional limit, it is found that the deflection increases less rapidly than would be expected from the earlier linear relationship between deflection and load. When the deflections are small the load is carried entirely by the bending stresses that are developed, that is, by compressive stresses on one side of the neutral plane and by tensile stresses on the other side. For moderately large values of the deflection, of the order of magnitude of the thickness of the plate, appreciable tensile (or in certain cases compressive) stresses are developed throughout the thickness of the plate. They are associated with the extension (or compression) of the material accompanying the deformation of the plate from its originally plane form. These stresses may be conveniently referred to as direct stresses. If they are tensile stresses the term membrane stresses is a very descriptive designation for them. The load is thus carried partly by the stiffness of the plate and partly by direct stresses that are developed in the middle surface and in surfaces parallel to it.

The determination of the deflection and the stress distribution of an isotropic plate with large deflections can be shown to depend upon the solution of two simultaneous partial differential equations of the fourth order.\textsuperscript{16} The maximum deflection is taken to be small in comparison with the length and breadth of the plates.

It is easy to modify the steps taken in the derivation of these equations for an isotropic plate\textsuperscript{17} to obtain the corresponding equations for the plywood plate. This is done in appendix 8. The equations obtained are (8.11) and (8.14). We shall have occasion to use them in determining the deflections and stresses of uniformly loaded infinite strips.

\textsuperscript{17} Hoo Nadai, A., Elastische Platten, pp. 284-287.
(practically, long narrow plates). The solution of these equations for finite rectangular plates has not been found. Considerations of energy will be used later to determine the approximate deflections at the center of such plates. However, the results obtained from the consideration of infinite strips will perhaps be found to have a wide range of application.

Even for isotropic plates solutions have not been found for the equations corresponding to (8.11) and (8.14) except in the cases of infinitely long strips and circular plates. 18

Section 9. Infinite Strip (Long Narrow Plate).
Large Deflections. Uniformly Distributed Load.
Edges Simply Supported

We shall now make use of the differential equations (8.11) and (8.14) to find the deflections and stresses of a uniformly loaded infinite strip with simply supported edges when the loads are such that the deflections are large and direct stresses are developed. It is assumed that there is no displacement of the edges associated with the direct stresses. This implies that the edges are restrained from moving in a direction perpendicular to the length of the plate. The solution obtained will be applicable in the central portion of a plate whose length is only moderately greater than its breadth. Approximate formulas are derived from a consideration of the exact formulas that are obtained for the infinite strip. These formulas are much simpler than the exact formulas and are sufficiently accurate for practical calculations. The approximate formulas here referred to are for the infinite strip. Approximate formulas will also be obtained in section 11 by the energy method for finite rectangular plates.

The exact solutions of equations corresponding to (8.11) and (8.14) were given for infinitely long plates of isotropic material with either simply supported or clamped edges by Stewart Way in a lithographed preprint of a paper presented to the American Society of Mechanical Engineers in 1932. It has apparently not been published in any other form. 19 From a reference in Way's paper it would appear that essentially the same solutions were given by I. Boobnoff in a book published in 1914 for the use of naval architects of the Russian Navy and not readily available in American libraries. The corresponding solutions for infinite strips of plywood are given below.

The edges of the strip are taken to be \( x = 0 \) and \( x = a \). Under the assumed uniform loading, the deflection and the strain and stress components are independent of \( y \). It can then be shown (see appendix 9) that the mean direct stress component \( X_x \) (the mean being taken over the thickness of the plate) is independent of \( x \) and is therefore a constant \( C \) for a given load \( p \). This information corresponds to that which would be

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18 See references to I. Boobnoff and S. Way in section 9.
19 The substance of this paper is found on pp. 4-17 of Timoshenko's Theory of Plates and Shells, 1940.
furnished by (8.11). The equation (8.14) becomes

\[ D \frac{d^4 w}{dx^4} = p + gh \frac{d^2 w}{dx^2} \]  

(9.5)

It is to be observed that \( g \) is a constant for a given load \( p \) but that it will have a different value when \( p \) is changed. The quantity \( g \) therefore enters the solution as a parameter whose value for a given load \( p \) must be determined in the course of solving the problem. A little consideration of the complications that arise in connection with the simple equation (9.5) will lead to an appreciation of the difficulties associated with the solution of the system of equations (8.11) and (8.14) in the general case.

The procedure to be followed in utilizing the solution of (9.5) subject to the conditions on the simply supported edges \( x = 0 \) and \( x = a \) to find the maximum bending stress, the direct stress, and the relation between deflection and load, is discussed in appendix 9.

However, in practical calculations it will not be necessary to follow this procedure, since it is possible to replace the exact formulas by quite accurate approximate formulas. With their aid the calculations involved in any given case are greatly simplified. These formulas whose derivation from the exact formulas is found in appendix 9 are the following:

(a) Relation between load and deflection

\[ p = A \frac{w_0}{h} + B \left( \frac{w_0}{h} \right)^3 \]  

(9.28)

where

\[ A = \frac{6.4\, E_h}{L} \]  

(9.29)

\[ B = \frac{30.6\, E_a}{L} \]  

(9.30)

(b) Maximum bending stress in a face ply

\[ s = \alpha \frac{E}{\lambda} \left( \frac{h}{a} \right)^2 \frac{w_0}{h} \]  

(9.31)
where \( \alpha \) may be taken to be 4.4. A method of obtaining a more accurate value of \( \alpha \) from a curve is explained below. The latter method is easy to apply and is to be preferred.

(c) Mean direct stress

\[
\sigma = 2.572 \frac{E_a(h)}{\lambda} \left( \frac{w_0}{h} \right)^2
\]

(9.25)

In these formulas \( E_1 \) denotes the mean modulus in bending, \( E_a \) the mean modulus in stretching, and \( E_X \) the actual modulus in a face ply, all measured parallel to the X-axis. The mean modulus in stretching \( E_a \) is merely the arithmetic mean of the \( E \)'s in the various plies measured in a direction parallel to the X-axis. Thus for three-ply plywood, having all plies of the same thickness and the grain of the face plies parallel to the X-axis

\[
E_a = \frac{2E_L + E_T}{3}
\]

In like manner \( E_b \) denotes the mean modulus in stretching in the direction parallel to the Y-axis.

The calculated values of the ratios \( E_a/E_L \) and \( E_b/E_L \) for various types of spruce and Douglas-fir plywood, using the values of \( E_L \) and \( E_T \) given in table 1 and on page 6, are shown in tables 29 and 30.

The grain of the face plies is taken to be parallel to the X-axis. If the grain of the face plies is parallel to the Y-axis, \( E_a/E_L \) and \( E_b/E_L \) as given in the tables are to be interchanged.

In formulas (9.25) to (9.31) \( \lambda \) may be taken to be 0.99. This is its value for spruce. For wood of other species this value may probably be used without appreciable error. It will be noted that the first term of (9.23) expresses the result obtained from the usual theory of thin plates when the deflections are assumed to be small.

Of the three formulas (9.23), (9.31), and (9.25), the second, namely, (9.31), is the least accurate if \( \alpha \) is taken to be 4.4. Actually \( \alpha \) ranges from 4.8 for small deflections to 4.0 for large deflections. Satisfactory values of \( \alpha \) can be readily obtained from the curve of figure 27 where \( \alpha \) is plotted as a function of the quantity \( \eta \) which is connected with the ratio \( w_0/h \) by the formula (see (9.24) appendix 9)

\[
\eta = 2.778 \left( \frac{E_a}{E_L} \right)^{1/2} \frac{w_0}{h}
\]
When \( \alpha \) is determined in this way the only approximation involved is that in the equation last written. This error is never large. In this connection see equation (9.24) of appendix 9 and the accompanying discussion.

The curve of figure 27 is plotted from the following data.

\[
\begin{array}{ccccccccccc}
\eta & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\alpha & 4.80 & 4.75 & 4.64 & 4.50 & 4.38 & 4.28 & 4.21 & 4.16 & 4.13 & 4.10 & 4.08 \\
\end{array}
\]

If the maximum bending stress, \( s \), is found by the method just described, it may be more convenient to calculate the mean membrane stress \( \tilde{g} \) from equation (9.18), appendix 9, instead of from (9.25). The value of \( \eta \) needed in (9.18) has been found in the calculation of \( s \).

From the mean direct stress and the maximum bending stress in a face ply as given by (9.18) and (9.20) or (9.25) and (9.31), the corresponding stresses in another ply can be found. The method of doing this is explained in appendix 9.

Useful formulas applicable to long narrow plates of isotropic material can be obtained from formulas (9.28), (9.31), and (9.25) by setting \( E_1 = E_a = E_x = E \) and \( \lambda = 1 - \sigma^2 \) where \( \sigma \) denotes Poisson's ratio for the material under consideration.

Tables 43 to 47 contain a comparison of the results obtained by using the approximate and exact formulas for various types of plates. At the time that the calculations for these tables were made, the use of the curve for \( \alpha \) had not been considered and \( \alpha \) was taken to be 4.4.

Section 10. Infinite Strip (Long Narrow Plate).
Uniformly Distributed Load. Edges Clamped.
Large Deflections

Exactly as in the case of an infinite strip with simply supported edges (see section 9 and appendix 9) exact formulas can be obtained for the case of an infinite strip with clamped edges and large deflections. From these formulas the deflection at the center, the mean direct stress, and the maximum bending stress can be calculated. These formulas are derived in appendix 10 where the tables necessary for their utilization are given. However, it is possible to replace the exact formulas by approximate formulas which are much simpler to use and are sufficiently accurate. The derivation of the approximate formulas is given in appendix 10. In this section as in section 9 the edges of the plate are restrained from moving inward.

The approximate formula connecting the load and the deflection is

\[
P = A \frac{w}{h} + B \left( \frac{w}{h} \right)^3
\]

(10.13)
where
\[ A = \frac{32}{\lambda} \frac{E_1}{E_L} \]  \hspace{1cm} (10.14)

\[ B = \frac{23}{\lambda} \frac{E_a}{E_L} \]  \hspace{1cm} (10.15)

and
\[ F = \frac{3E}{E_L h} \]  \hspace{1cm} For the definitions of \( E_1, E_a, \lambda \) see section 9 or the table of notations. For the maximum bending stress in a face ply we have the approximate formula

\[ s = C \frac{V_0}{h} + D \left( \frac{V_0}{h} \right)^3 \]  \hspace{1cm} (10.20)

where
\[ C = \frac{16E_X}{\lambda} (\frac{h}{a})^2 \]  \hspace{1cm} (10.21)

\[ D = \frac{2.98E_X}{\lambda} \frac{E_a}{E_L} (\frac{h}{a})^2 \]  \hspace{1cm} (10.22)

The symbol \( E_X \) denotes the Young's modulus in a face ply and in the direction parallel to the \( X \)-axis.

The range of values of \( \frac{V_0}{h} \) within which this formula may be used is discussed in appendix 10.

A better approximation to \( s \) is obtained from the formula

\[ s = \alpha \frac{E_X}{\lambda} (\frac{h}{a})^2 \frac{V_0}{h} \]  \hspace{1cm} (10.16)

and the curve of figure 28 where the argument \( \eta \) is connected with \( \frac{V_0}{h} \) by the equation

\[ \frac{V_0}{h} = 0.366 \left( \frac{E_1}{E_a} \right)^{1/2} \eta \]  \hspace{1cm} (10.10)

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The curve of figure 28 can be plotted from the following data:

\[
\begin{array}{cccccccccccc}
\eta & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\alpha & 16.00 & 16.52 & 18.03 & 20.31 & 23.20 & 26.44 & 29.92 & 33.58 & 37.30 & 41.15 & 45.00
\end{array}
\]

If this method is used to find \( \sigma \) it may be more convenient to calculate the direct stress \( \sigma \) from (10.7) of appendix 10 instead of using (10.11) below.

The approximate formula for the mean direct stress is

\[
\sigma = \frac{2.4 s E a}{h} \left( \frac{h}{a} \right)^2 \left( \frac{r}{h} \right)^2
\]

(10.11)

From the direct stress and the maximum bending stress in a face ply as given by (10.11) and (10.20) the corresponding stress in any given ply can be calculated by the method explained in appendix 9. See equations (9.33) to (9.36).

Tables 48 to 52 in appendix 10 show comparisons of the results of calculations made with the exact and the approximate formulas. In the tables the maximum bending stress was calculated by formula (10.20) instead of by the more accurate procedure based on equation (10.16) and the curve of figure 28.

These formulas are also applicable to long narrow isotropic plates. It is only necessary to use \( E \) in place of all letters \( E \) that have subscripts and take \( \lambda \) equal to \( 1 - \sigma^2 \) where \( \sigma \) is Poisson's ratio.

Section 11. Rectangular Plate. Uniformly Distributed Load. Edges Simply Supported. Approximate Method

The solution of the differential equations (8.11) and (8.14) that describe the behavior of a flat plate when the deflections are large, has not been found for the rectangular plate because of the mathematical difficulties associated with the fact that these equations are not linear.

To obtain an approximate expression for the deflection at the center of the plates considerations of energy are employed as was done in

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The analysis of sections 11 and 12 and appendixes 11 and 12 was carried out by the author during a semester in which he was relieved from teaching duties under a grant to the University of Wisconsin from the Wisconsin Alumni Research Foundation. The numerical calculations were made by the Computing Division of the Forest Products Laboratory.
sections 3 and 4 in obtaining approximate formulas in the case of plates with small deflections. In the present case since the middle surface of the plate is in a state of strain it is necessary to assume suitable expressions for the components \( u \) and \( v \) parallel to the \( X \) and \( Y \) axes, respectively, of the displacement of points in the middle surface of the plate, in addition to a suitable expression for the deflection. These expressions contain certain parameters which are to be chosen in such a way that the total potential energy of the plate and applied load is a minimum.

The potential energy of deformation is considered to be made up of two parts, that of the state of strain associated with the bending stresses and that of the state of strain associated with the direct stresses. This implies that both states of strain are considered to be so small that the potential energy of the sum of the two states of strain is approximately equal to the sum of their respective potential energies.

In the expressions assumed for the deflection and the displacement there are four parameters which are to be determined so that the total potential energy is a minimum. It is found that the determination of one of these parameters, \( \tau \) (see appendix 11), in this way involves calculations that are too complicated. Accordingly, \( \tau \) is taken to be the same as in the case of small deflections. This determines the general shape of the assumed deformed middle surface when the deflections are small. The subsequent argument, for the case of large deflections, rests upon the assumption that the form of the middle surface does not change greatly as the deflections become larger, all ordinates of the middle surface for a small deflection being considered to be multiplied by a common factor of proportionality.

The parameter \( \tau \) having been chosen in this way, the remaining parameters can be found. After performing the calculations (see appendix 11) the following formula is found connecting the quantity

\[
P = \frac{E_1 h^4}{p a^4}
\]

and the ratio \( \frac{w_o}{h} \) of the deflection at the center to the thickness:

\[
P = H \frac{w_o}{h} + Q \left( \frac{w_o}{h} \right)^3
\]  

(11.23)

The factor \( H \) is the reciprocal of the factor \( \alpha \) in (3.37), the corresponding formula for a plate with small deflections. This is to be expected, since from the way in which it was derived (11.23) must agree with (3.37) when \( \frac{w_o}{h} \) is small. A formula for calculating the factor \( Q \) is found in
appendix 11. Tables 31 to 35 give the factors $H$ and $Q$ for plates of source plywood of several types and for plates of isotropic material. It is to be recalled that the plies are assumed to be of equal thickness in a given plate and that the elastic constants of the wood are those given in table 1. The letter $k$ denotes the ratio $b/a$ of the sides of the rectangle, the side $b$ being parallel to the $Y$-axis. It is to be borne in mind that formula (11.23) is an approximate one and that the errors involved in using it may be considerable. However, it is believed that the formula will give a reasonable estimate of the deflection associated with a given load for a given plate. If the length of a plywood plate is sufficiently greater than the breadth so that the ratio

$$\beta/a = (b/a)/(E_1E_2)^{1/4}$$

is greater than 2, the curve of figure 11 indicates that the plate may be considered to be a long narrow plate. Then either the approximate or exact methods explained in section 9 can be used. The approximate method explained in that section will be found to be sufficiently accurate. For such values of the ratio $\beta/a$, the approximate method explained in section 9 can also be used to find the maximum bending stress and the direct stress. The ratio $\beta/a$ is to be distinguished from the ratio $k = b/a$ of tables 31 to 35.

It is interesting to compare the approximate formula for the infinite strip as found for the limiting case $k \rightarrow \infty$ by the methods of this section with formula (9.28) which represents approximately the results of the exact theory. For the infinite strip from (11.24) and (11.25) (see appendix 11),

$$P = \frac{6.32}{\lambda} x_1 \frac{W_0}{h} + \frac{19.12}{\lambda} x_4 \left( \frac{W_0}{h} \right)^2$$

(11.26)

while from (9.28)

$$P = \frac{6.4}{\lambda} x_1 \frac{W_0}{h} + \frac{20.6}{\lambda} x_4 \left( \frac{W_0}{h} \right)^2$$

(11.27)

Section 12. Rectangular Plate. Uniformly Distributed Load. Edges Clamped. Large Deflections. Approximate Method

The method of section 11 will be applied to find an approximate expression for the deflection at the center of uniformly loaded plates with clamped edges.

The following forms are assumed for the deflection and the components of the displacement:
When \( 0 < y < c \)

\[
\begin{align*}
  w &= w_0 \sin^2 \frac{\pi y}{2c} \sin^2 \frac{\pi x}{a}, \\
  u &= c_1 \sin \frac{\pi y}{2c} \sin \frac{4 \pi x}{a}, \\
  v &= c_2 \sin \frac{2 \pi y}{c} \sin \frac{\pi x}{a};
\end{align*}
\]

(12.1)

when \( c < y < b-c \),

\[
\begin{align*}
  w &= w_0 \sin^2 \frac{\pi x}{a}, \\
  u &= c_1 \sin \frac{4 \pi x}{a}, \\
  v &= 0.
\end{align*}
\]

(12.2)

Deflections and displacements corresponding to those given by (12.1) are assumed for the region \( b-c < y < b \). The forms (12.2) were chosen to represent approximately the situation in an infinite strip, \( w \) being chosen to satisfy the conditions at the edges and \( u \) in such a way that the mean direct stress \( X' \) is constant. The forms (12.1) were then chosen to satisfy the conditions at the edges of the plate and to pass over continuously into the forms (12.2) along the line \( y = c \).

The following formula (see appendix 12) is found connecting the load and the deflection at the center:

\[
P = H \frac{w_0}{h} + Q \left( \frac{w_0}{h} \right)^3
\]

(12.3)

where

\[
P = \frac{2 \pi^3 h^4}{L_1 L_2 L_3}
\]

and \( H \) and \( Q \) are defined by (12.4) and (12.5) in appendix 12.
The factor $H$ in (12.3) is the reciprocal of the factor $\alpha$ in (4.35). That this should be so follows at once from the way in which the formulas were obtained. The remarks made concerning the approximate character of equation (11.23) also apply to equation (12.3). The curve of figure 19 indicates that if the ratio $\beta/a = (b/a) \left( \frac{E_1}{E_2} \right)^{1/4}$ is greater than 1.75, the central portion of the plate can be treated as part of a long narrow plate. The methods of section 10 can then be employed to determine not only the deflections but also the bending and direct stresses in the central portion of the plate.

For isotropic rectangular plates with large deflections an approximate treatment has recently been given by S. Way\textsuperscript{21}\ using expressions different from (12.1) and (12.2) and containing a larger number of parameters. He obtained curves connecting the load and the maximum deflection and also curves connecting the load and the maximum stress, for three rectangles, the ratios of whose sides are 1, 3/2, and 2, respectively. Because of the larger number of parameters employed his results undoubtedly represent a better approximation to the actual solution than those based on equations (12.1) and (12.2) although the amount of numerical calculation is much greater with the increased number of parameters. For isotropic plates the loads associated with a given deflection calculated by the two methods differ by less than 12 percent, usually by much less, for deflections in which $\frac{v_0}{h}$ lies between 0.5 and 2.

No attempt has been made to calculate the maximum stresses on the basis of equations (12.1) and (12.2). It is probable that equation (12.3) yields values of $P$ for which the error is of the order of magnitude of 10 percent. The labor involved in obtaining more accurate values of $P$ for each type of plywood plate by the energy method is prohibitive. As noted above, the methods of section 10 can be expected to yield satisfactory values of the deflection and stress in the central portion of plates for which $\beta/a = (b/a) \left( \frac{E_1}{E_2} \right)^{1/4}$ is greater than 1.75.

Tables 36 to 40 give the values of $H$ and $Q$ for several types of spruce plywood, the plies being assumed to be of equal thickness.

There is close agreement between the approximate formula (12.3) for the case $k = b/a = \infty$ and the approximate formula (10.13) for the infinite strip derived in appendix 10 as an approximation from the exact formulas. The formula (10.13) was

$$P = \frac{32}{\lambda} \times \frac{v}{h} + \frac{23}{\lambda} \times a \left( \frac{w_0}{h} \right)^3$$

while formula (12.3) becomes, for $k = \infty$,

$$P = \frac{32.5}{\lambda} \times \frac{w_0}{h} + \frac{24.4}{\lambda} \times a \left( \frac{v_0}{h} \right)^3$$


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Appendix I.—Stress-strain Relations in an Orthotropic Material

The energy of deformation of an orthotropic material can be written in the form:

\[ 2W = Ae_{xx}^2 + Be_{yy}^2 + Ce_{zz}^2 \]
\[ + 2Fe_{yy}e_{zz} + 2Ge_{zz}e_{xx} + 2He_{xx}e_{yy} \]
\[ + Le_{yz}^2 + Me_{zx}^2 + Ne_{xy}^2, \]

the coordinate planes being parallel to the planes of elastic symmetry. It is not necessary to discuss the significance of the coefficients \( A, B, C \) ... in equation (1.1). It is sufficient to remark that they are numbers that characterize the elastic behavior of the material. The usual elastic moduli are introduced in equations (1.5) and their relations to the numbers \( A, B, C \) etc., are shown in equations (1.4), (1.6), and (1.7).

The relations between the components of stress and strain are obtained from the equations

\[ X_x = \frac{\partial W}{\partial e_{xx}}, \ldots \ldots \quad X_y = \frac{\partial W}{\partial e_{xy}} \]

Then

\[ X_x = Ae_{xx} + He_{yy} + Ge_{zz} \]
\[ Y_y = He_{xx} + Be_{yy} + Fe_{zz} \]
\[ Z_z = Ge_{xx} + Fe_{yy} + Ce_{zz} \]

\[ \text{See for example: Love, A. E. H., The Mathematical Theory of Elasticity, Art. 110; St. Venant in his annotated translation of Clebsch, Théorie de l'Elasticité des Corps Solides, pp. 76-80; Price, A. T., Phil. Trans. 228A, 1-62, 1928. Love's notation for the components of stress and strain will be used throughout.} \]
\[ Y_z = L e_{yz}, \quad Z_x = M e_{zx}, \quad X_y = N e_{xy} \] (1.4)

The solution of the equations (1.3) for the strain components in terms of the stress components can be written in the form

\[
\begin{align*}
\varepsilon_{xx} &= \frac{1}{E_x} X_x - \frac{\sigma_{yx}}{E_y} Y_y - \frac{\sigma_{zx}}{E_z} Z_z \\
\varepsilon_{yy} &= -\frac{\sigma_{xy}}{E_x} X_x + \frac{1}{E_y} Y_y - \frac{\sigma_{zy}}{E_z} Z_z \\
\varepsilon_{zz} &= -\frac{\sigma_{xz}}{E_x} X_x - \frac{\sigma_{yz}}{E_y} Y_y + \frac{1}{E_z} Z_z
\end{align*}
\] (1.5)

where, with \( \Delta \) denoting the determinant of the coefficients of (1.3)

\[
\frac{1}{E_x} = \frac{BC - F^2}{\Delta}, \quad \frac{1}{E_y} = \frac{CA - G^2}{\Delta}, \quad \frac{1}{E_z} = \frac{AB - H^2}{\Delta}
\] (1.6)

\[
\frac{\sigma_{yx}}{E_y} = \frac{\sigma_{xy}}{E_x} = \frac{CH - FG}{\Delta}
\] (1.7)

There are two further equations that can be written down by cyclic permutation of the letters in (1.7).

We observe accordingly that there are three Young's moduli, \( E_x, E_y, \) and \( E_z \); six Poisson's ratios \( \sigma_{xy}, \sigma_{yx}, \sigma_{yz}, \sigma_{zy}, \sigma_{zx}, \sigma_{xz} \). From (1.4) it follows that there are three shearing moduli \( \mu_{yz} = L, \mu_{zx} = M, \) and \( \mu_{xy} = N \). Among these twelve constants there are three relations, namely, those expressed by (1.7) and two similar equations.
Appendix 2. The Differential Equation for the
Deflection of a Plywood Plate. Small Deflections

In addition to the assumptions explicitly stated in section 2, it is assumed as is usual in the theory of thin plates that the points of a straight line which is normal to the undeformed plane middle surface, \( Z = 0 \), of the plate, remain in a straight line which is normal to the middle surface after deformation has taken place. The deflections are assumed to be so small that direct stresses (see section 6) are not developed to an appreciable extent.

Under these assumptions and with the choice of axes shown in figure 1, the components, \( U \) and \( V \), parallel to the \( X \)- and \( Y \)-axes, respectively, of the displacement of a point whose coordinate with respect to the middle plane is \( Z \) are expressed\(^{23}\) by the equations:

\[
U = -Z \frac{\partial W}{\partial x} \\
V = -Z \frac{\partial W}{\partial y}
\]

(2.1)

where \( W \) denotes the deflection of a point in the middle surface. From these equations the strain components are found to be

\[
e_{xx} = -Z \frac{\partial^2 W}{\partial x^2}, \quad e_{yy} = -Z \frac{\partial^2 W}{\partial y^2}, \quad e_{xy} = -2Z \frac{\partial^2 W}{\partial x \partial y}
\]

(2.2)

The stress component \( Z \) is taken to be negligible in comparison with \( X_x \) and \( Y_y \). It follows from (1.5) that at a point in a given ply

\[
X_x = \frac{E_x}{\lambda} \left( e_{xx} + \sigma_{yx} e_{yy} \right) \\
Y_y = \frac{E_y}{\lambda} \left( e_{yy} + \sigma_{xy} e_{xx} \right) \\
X_y = \mu_{xy} e_{xy}
\]

(2.3)

where

\[
\lambda = 1 - \sigma_{xy} \sigma_{yx}
\]

(2.4)

\(^{23}\) See, for example, Nadai, A., Elastische Platten, (Berlin 1925) p. 19.
Hence, using (2.2)

$$X_x = -\frac{E_x z}{\lambda} \left( \frac{\partial^2 W}{\partial x^2} + \sigma_{yx} \frac{\partial^2 W}{\partial y^2} \right)$$

$$Y_y = -\frac{E_y z}{\lambda} \left( \frac{\partial^2 W}{\partial y^2} + \sigma_{xy} \frac{\partial^2 W}{\partial x^2} \right)$$

(2.5)

$$X_y = -2\mu_{xy} z \frac{\partial^2 W}{\partial x \partial y}$$

The differential equation for the deflection \(W\) is readily obtained\(^{24}\) from the conditions for the equilibrium of an element of the plate (see figure 29).

The bending moments are denoted by \(M_x\) and \(M_y\), the twisting moment by \(M_{xy}\), and the vertical shearing forces by \(p_x\) and \(p_y\), all measured per unit length of the edge of the elements along which they act.

The moments \(M_x\), \(M_y\), and \(M_{xy}\) are defined by the following equations:

$$m_x = \int_{-h/2}^{h/2} X_x z \, dz, \quad m_y = \int_{-h/2}^{h/2} Y_y z \, dz$$

(2.6)

$$m_{xy} = \int_{-h/2}^{h/2} X_y z \, dz$$

The moments acting on a small rectangular element of the plate are represented by vectors in figure 29.

From (2.6), using (2.5)

\(^{24}\) For the corresponding treatment of the isotropic plate, see Nadai, A., Elastische Platten, p. 20.
\[ m_x = -a_1 \frac{\partial^2 W}{\partial x^2} - a_2 \frac{\partial^2 W}{\partial y^2} \]
\[ m_y = -b_1 \frac{\partial^2 W}{\partial x^2} - b_2 \frac{\partial^2 W}{\partial y^2} \]
\[ m_{xy} = c \frac{\partial^2 W}{\partial x \partial y} \]

(2.7)

where
\[ a_1 = \int_{-h/2}^{h/2} \left( E_x z^2 / \lambda \right) dz , \quad a_2 = \int_{-h/2}^{h/2} \left( E_x \sigma_{yx} z^2 / \lambda \right) dz , \]
\[ b_1 = \int_{-h/2}^{h/2} \left( E_y \sigma_{xy} z^2 / \lambda \right) dz , \quad b_2 = \int_{-h/2}^{h/2} \left( E_y z^2 / \lambda \right) dz , \]
\[ c = 2 \int_{-h/2}^{h/2} \mu_{xy} z^2 dz \]

(2.8)

From the relation (1.7) it follows that \( a_2 = b_1 \).

The vertical shearing forces \( p_x \) and \( p_y \) are defined by the following equations:
\[ p_x = \int_{-h/2}^{h/2} Z_x dz , \quad p_y = \int_{-h/2}^{h/2} Z_y dz \]

(2.9)

They are represented by vectors in figure 4, c.

The conditions for the equilibrium of moments with respect to the X-axis, and Y-axis, respectively, lead to the equations:
\[ p_x = \frac{\partial m_x}{\partial x} + \frac{\partial m_{xy}}{\partial y} \]  \hspace{1cm} (2.10)

\[ p_y = \frac{\partial m_y}{\partial y} + \frac{\partial m_{xy}}{\partial x} \]

while the condition for the equilibrium of forces acting in the direction of the Z-axis leads to the equation

\[ \frac{\partial p_x}{\partial x} + \frac{\partial p_y}{\partial y} + p = 0 \]  \hspace{1cm} (2.11)

It is to be recalled from section 2 that \( h \) denotes the thickness of the plate and \( p \) the load per unit area on the face \( z = -h/2 \).

From (2.7), (2.10), and (2.11) the following differential equation for the deflection, \( W \), is found:

\[ D_1 \frac{\partial^4 W}{\partial x^4} + 2K \frac{\partial^4 W}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 W}{\partial y^4} = p \]  \hspace{1cm} (2.12)

where

\[ D_1 = a_4, \quad D_2 = b_2, \quad K = \frac{1}{2} (a_2 + b_1 + 2c) \]  \hspace{1cm} (2.13)

If the plies in the plate are all flat grained, as assumed, the expressions for \( D_1, D_2 \) and \( K \) can be simplified. The subscripts \( _L \) and \( _T \) being used to refer to the longitudinal and tangential directions in the wood, it is clear that for plies in which the grain of the wood is parallel to the X-axis

\[ E_x = E_L, \quad E_y = E_T, \quad \mu_{xy} = \mu_{LT}, \]  \hspace{1cm} (2.14)

\[ \sigma_{xy} = \sigma_{LT}, \quad \sigma_{yx} = \sigma_{TL} \]

while for plies in which the grain of the wood is parallel to the Y-axis

\[ E_x = E_T, \quad E_y = E_L, \quad \mu_{xy} = \mu_{TL} = \mu_{LT}, \]  \hspace{1cm} (2.15)

\[ \sigma_{xy} = \sigma_{TL}, \quad \sigma_{yx} = \sigma_{LT} \]
The coefficients $D_1$ and $D_2$ of (2.12) are readily expressed
in terms of certain mean moduli in bending $E_1$ and $E_2$. Thus we write

$$D_1 = \frac{E_1 h^3}{12 \lambda} , \quad D_2 = \frac{E_2 h^3}{12 \lambda} \quad (2.16)$$

where

$$\lambda = 1 - \frac{\sigma_{LT}}{\sigma_{TL}} \quad (2.17)$$

$$E_1 = \frac{12}{h^3} \int_{-h/2}^{h/2} E_x z^2 \, dz = \frac{12}{h^3} \int_{-h/2}^{h/2} E_x z^2 \, dz \quad (2.18)$$

$$E_2 = \frac{12}{h^3} \int_{-h/2}^{h/2} E_y z^2 \, dz \quad (2.19)$$

The quantities $E_1$ and $E_2$ may be called the "mean moduli" in bending" under couples whose axes are perpendicular to the $XZ$ and $YZ$ planes, respectively. As soon as the structure of the plywood is known these moduli are readily calculated in terms of the Young's moduli $E_L$ and $E_T$ of the wood in question.

For a plate whose construction is not symmetrical with respect to the middle plane, it is to be expected that $E_1$ and $E_2$, as defined on page 7, may be used with slight error in the formulas of this report, although these formulas were derived from an analysis that assumed a symmetrical construction of the plate. The quantities $E_1$ and $E_2$ determine the stiffness of the plate in the two principal directions. The term in the differential equation involving shear is independent of the situation as to symmetry so long as all the plies are flat grain (or all edge grain).

Since the plies are all assumed to be flat grained and since, in accordance with (1.7)

$$E_L \sigma_{LT} = E_T \sigma_{LT} \quad (2.20)$$

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25 Price, A. T., Phil. Trans. A 228, 1, 1928. It is pointed out in a footnote in section 2 that the definition of these moduli there used differs somewhat from that given by Price.
it follows that the factors

\[ E_x \frac{\sigma_{yx}}{\lambda} \quad \text{and} \quad E_y \frac{\sigma_{xy}}{\lambda} \]

in the expressions for \( a_2 \) and \( b_1 \) in (2.8) are the same for all plies and are equal to \( E_L \sigma_{TL}/\lambda \)

Hence

\[ a_2 = b_1 = \frac{E_L \sigma_{TL}}{\lambda} \frac{h^3}{12} \quad (2.21) \]

and

\[ K = \left( \frac{E_L \sigma_{TL}}{\lambda} + 2 \mu_{LT} \right) \frac{h^3}{12}, \quad (2.22) \]

\( \mu_{xy} \) being identical with \( \mu_{LT} \) for all plies.

The expressions (2.7) for the bending and twisting moments can now be written in the forms:

\[ m_x = -D_1 \left( \frac{\partial^2 w}{\partial x^2} + \sigma_1 \frac{\partial^2 w}{\partial y^2} \right) \quad (2.23) \]

\[ m_y = -D_2 \left( \frac{\partial^2 w}{\partial y^2} + \sigma_2 \frac{\partial^2 w}{\partial x^2} \right) \]

\[ m_{xy} = -\frac{\mu_{LT} h^3}{6} \frac{\partial^2 w}{\partial x \partial y} \]

where

\[ \sigma_1 = \frac{E_L \sigma_{TL}}{E_1}, \quad \sigma_2 = \frac{E_L \sigma_{TL}}{E_2} \quad (2.24) \]

The differential equation (2.12) can be reduced to the simpler form

\[ \frac{\partial^4 w}{\partial x^4} + 2 \kappa \frac{\partial^4 w}{\partial x^2 \partial \eta^2} + \frac{\partial^4 w}{\partial \eta^4} = \frac{p}{D_1} \quad (2.25) \]
where

\[ \kappa = K / (D_1 D_2)^{1/2} \]  (2.26)

by making the substitution

\[ \eta = \varepsilon y \]  (2.27)

where

\[ \varepsilon = \left( D / D_2 \right)^{1/4} = \left( E / E_2 \right)^{1/4} \]  (2.28)

The change of variables from \( X \) and \( Y \) to \( X \) and \( \eta \) corresponds to a simple extension or contraction of the plate parallel to the \( Y \)-axis in the ratio \( \varepsilon : 1 \). A plate of dimensions \( a \) and \( b \) is thus transformed into one of dimensions \( \alpha \) and \( \beta = \varepsilon b = b \left( E / E_2 \right)^{1/4} \).

The latter plate will be referred to as the transformed plate. In the new variables the expressions (2.23) for the bending and twisting moments become:

\[ m_x = - D_1 \left( \frac{d^2 w}{d x^2} + \sigma_1 \varepsilon^2 \frac{d^2 w}{d \eta^2} \right) \]

\[ m_y = D_2 \left( \varepsilon^2 \frac{d^2 w}{d \eta^2} + \sigma_2 \frac{d^2 w}{d x^2} \right) \]  (2.29)

\[ m_{xy} = - \frac{\mu_{LT}}{6} \varepsilon h^3 \frac{d^2 w}{d x d \eta} \]
Appendix 3.--Rectangular Plate Under Uniform Load, Edges Simply Supported. Small Deflections

The differential equation (2.25)

$$\frac{\partial^4 W}{\partial x^4} + 2\kappa \frac{\partial^4 W}{\partial x^2 \partial \eta^2} + \frac{\partial^4 W}{\partial \eta^4} = \frac{p}{D_i}$$

where $p$ is a constant, is to be solved subject to the conditions stated below that hold on the edges $x = 0$, $x = a$, $\eta = 0$, $\eta = \beta$ where (See (2.28))

$$\beta = \varepsilon b$$

(3.1)

The boundary conditions on the edges $x = 0$ and $x = a$ are (See (2.29))

$$W = 0, \quad \frac{\partial^2 W}{\partial x^2} + \sigma \varepsilon^2 \frac{\partial^2 W}{\partial \eta^2} = 0$$

(3.2)

The corresponding conditions on the edges $\eta = 0$ and $\eta = \beta$ (that is, $y = b$) are

$$W = 0, \quad \varepsilon^2 \frac{\partial^2 W}{\partial \eta^2} + \sigma_2 \frac{\partial^2 W}{\partial x^2} = 0$$

(3.3)

For the constants $\sigma_1$, $\sigma_2$ and $\varepsilon$ see equations (2.24) and (2.28).

We choose first the following solution of equation (2.25):

$$W_i = \frac{p}{24D_i} \left( x^4 - 2ax^3 + a^3x \right)$$

(3.4)

This solution satisfies the boundary conditions (3.2). It represents, in fact, the deflection of a uniformly loaded infinitely long strip of plywood having its edges, $x = 0$ and $x = a$, simply supported.

It will be convenient to write this solution in the form

$$W_i = A \sum_{n} \frac{1}{\lambda_n^3} \sin \lambda_n x, \quad n = 1, 3, 5, \ldots$$

(3.5)
where

$$A = 4 \rho / a D,$$  \hspace{1cm} (3.6)$$

and

$$\lambda_n = n \pi / a$$  \hspace{1cm} (3.7)$$

The satisfaction of the boundary conditions on the edges \( \eta = 0 \) and \( \eta = \beta \) will be secured by combining with \( \mathcal{W}_1 \), a solution, \( \mathcal{W}_2 \), of the equation (2.25) with its right hand member set equal to zero. Let

$$\mathcal{W}_2 = - \sum_{n=1}^{\infty} \frac{A}{\lambda_n^6} Y_n \sin \lambda_n x$$  \hspace{1cm} (3.8)$$

where \( Y_n \) is a function of \( \eta \). In order that \( \mathcal{W}_2 \) may satisfy (2.25) with its right hand member set equal to zero, \( Y_n \) must satisfy the differential equation

$$Y_n^{IV} - 2 \kappa \lambda_n^2 Y_n^{II} + \lambda_n^4 Y_n = 0$$

On setting \( Y_n = e^{m_n \eta} \) it is readily found that

$$m_n = \pm \gamma_n \pm i \delta_n$$

where

$$\gamma_n = \lambda_n \rho \hspace{1cm} \delta_n = \lambda_n \sigma$$  \hspace{1cm} (3.9)$$

$$\rho = \sqrt{\frac{1+\kappa}{2}} \hspace{1cm} \sigma = \sqrt{\frac{1-\kappa}{2}}$$  \hspace{1cm} (3.10)$$

Then \( Y_n \) may be chosen as a suitable linear combination of the following functions:

$$\sinh \gamma_n \eta \sin \delta_n \eta, \hspace{0.5cm} \sinh \gamma_n \eta \cos \delta_n \eta$$  \hspace{1cm} (3.11)$$

$$\cosh \gamma_n \eta \sin \delta_n \eta, \hspace{0.5cm} \cosh \gamma_n \eta \cos \delta_n \eta$$
In obtaining these solutions it has been assumed that \( \kappa \) is less than 1. This appears to be true for all types of plywood. If \( \kappa \geq 1 \) appropriate modifications in the functions (3.11) can be made.

It is clear that \( \mathcal{W}_2 \) satisfies the conditions (3.2). It remains to choose the coefficients of a linear combination of the foregoing solutions (3.11) in such a way that \( \mathcal{W}_1 + \mathcal{W}_2 \) satisfies the conditions (3.3). It is found that when \( n \) is odd

\[
\mathcal{Y}_n = C_n \left\{ \kappa \left[ \sinh \nu_n \eta \sin \delta_n (\beta - \eta) + \sinh \nu_n (\beta - \eta) \sin \delta_n \eta \right] \\
+ \sqrt{1 - \kappa^2} \left[ \cosh \nu_n \eta \cos \delta_n (\beta - \eta) \\
+ \cosh \nu_n (\beta - \eta) \cos \delta_n \eta \right] \right\}
\]

where

\[
C_n = \frac{1}{\sqrt{1 - \kappa^2} (\cosh \nu_n \beta + \cos \delta_n \beta)}
\]

(3.12)

and that when \( n \) is even

\[
\mathcal{Y}_n = 0
\]

Then the deflection is given by

\[
\mathcal{W} = \mathcal{W}_1 + \mathcal{W}_2
\]

\[
= \sum_n \frac{A_n}{\lambda_n^5} (1 - \mathcal{Y}_n) \sin \lambda_n x
\]

\[
= \frac{4p \alpha^4}{\pi^5 D_1} \sum_n \frac{1}{n^5} (1 - \mathcal{Y}_n) \sin \lambda_n x, \quad n = 1, 3, 5, \ldots
\]

(3.14)

On recalling that \( D_1 = E_1 h^3/12 \lambda \) equation (3.14) may be written

\[
\frac{\mathcal{W}}{h} = \frac{48 \lambda}{\pi^5} \frac{E_1}{E_i} \rho \sum_n \frac{1}{n^5} (1 - \mathcal{Y}_n) \sin \lambda_n x
\]

\[
\quad n = 1, 3, 5, \ldots
\]

(3.15)
where

\[ P = \frac{p \alpha^4}{E_L h^4}, \quad \lambda = 1 - \sigma_T \sigma_L \]  \hspace{1cm} (3.16)

Using (3.15) and (2.29) expressions can readily be found for the deflections and moments at the center or at any other point of the plate. At the center of the plate the ratio of the deflection \( W_0 \) to the thickness \( h \) is found to be given by the equation,

\[
\frac{W_0}{h} = \frac{48 \lambda}{\pi^5} \frac{E_L}{E_t} P \sum \frac{(-1)^n}{n^5} \left[ 1 - 2C_n (\kappa \sinh \frac{\gamma_n \beta}{2} \sin \frac{\delta_n \beta}{2} 
+ \sqrt{1 - \kappa^2} \cosh \frac{\gamma_n \beta}{2} \cos \frac{\delta_n \beta}{2} \right] \]  \hspace{1cm} (3.17)

\( n = 1, 3, 5 \ldots \)

An approximate formula will now be obtained for the deflection at the center by assuming a plausible form for the deflected middle surface and determining certain parameters that appear in this assumed form, in such a way that the sum of the potential energy of deformation of the plate and that of the applied load shall be a minimum. (See discussion in section 3.)

Let (See figures 4 and 5, section 3.)

\[ W = W_0 \sin \frac{\pi x}{\alpha} \]

when \( c < y < b - c \) \hspace{1cm} (3.18)

\[ W = W_0 \sin \frac{\pi x}{\alpha} \sin \frac{\pi y}{2c} \]

when \( 0 < y < c \)

A form corresponding to the latter will be assumed for the portion of the plate for which \( b - c < y < b \) but it need not be written down since the potential energy of the plate can be calculated as twice the potential energy of the portion of the plate for which

\[ 0 < y < \frac{b}{2} \]

Let

\[ c = \pi a / 2 \]  \hspace{1cm} (3.19)
The deflection \( W \) is expressed as a function of the two parameters \( W_0 \) and \( \gamma \) which are to be determined in such a way that the total potential energy of the system is a minimum. Let

\[
V_t = \text{change in the potential energy of the load due to the deflection.}
\]

\[
V_{be} = \text{potential energy of deformation of the portions of the plate at the ends.}
\]

\[
V_{bm} = \text{potential energy of deformation of the middle portion of the plate.}
\]

\[
V_b = V_{be} + V_{bm}
\]

Now

\[
V_t = -p \int_{-h}^{h} \int_{0}^{a} \int_{0}^{b} \int_{0}^{b} W \, dx \, dy \quad (3.20)
\]

It is known that the potential energy of deformation is given by

\[
V_b = \frac{1}{2} \int_{-h}^{h} \int_{0}^{a} \int_{0}^{b} (X_x e_{xx} + Y_y e_{yy} + X_y e_{xy}) \, dz \, dx \, dy
\]

We substitute for \( X_x, Y_y, \) and \( X_y \) their values as given by (2.3) in terms of the strain components \( e_{xx}, e_{yy}, \) and \( e_{xy} \). The integrand is then a quadratic function of these strain components. For the strain components we then substitute their values in terms of \( \frac{\partial^2 W}{\partial x^2}, \frac{\partial^2 W}{\partial y^2}, \) and \( \frac{\partial^2 W}{\partial x \partial y} \) as given by (2.2) and perform the integration with respect to \( z \). We thus obtain

\[
V_b = \frac{h^2 E_I}{24 \lambda} \int_{0}^{a} \int_{0}^{b} \left\{ \kappa_x \left( \frac{\partial^2 W}{\partial x^2} \right)^2 + \kappa_z \left( \frac{\partial^2 W}{\partial y^2} \right)^2 \right. \\
\left. + 2 \sigma_t \frac{\partial^2 W}{\partial x^2} \frac{\partial^2 W}{\partial y^2} + 4 \lambda v \left( \frac{\partial^2 W}{\partial x \partial y} \right)^2 \right\} \, dx \, dy \quad (3.21)
\]
where

\[ \kappa_1 = \frac{E_h}{E_L}, \quad \kappa_2 = \frac{E_h}{E_L}, \quad \nu = \frac{\mu_{LT}}{E_L} \]  \hspace{1cm} (3.22)

Since the definition of \( W \) in (3.21) is different in the middle and end portions of the plate it is simpler to calculate separately the potential energy of deformation of these portions.

The potential energy \( V_{bm} \) is obtained by integrating the same expression as in (3.21) over the central portion of the plate, \( W \) being there defined by the first of the expressions in (3.18). In like manner \( V_{be} \) is found by taking twice the result of extending the integration over the portion of the plate between the lines \( y = 0 \) and \( y = C \), \( W \) being defined by the second of the expressions in (3.18).

Using the abbreviation

\[ k = \frac{b}{a} \]  \hspace{1cm} (3.23)

we find from (3.18), (3.20) and (3.21) that

\[ V_{el} = - \frac{4 \pi^2 \lambda W_0}{\pi^2} \left[ \beta + \frac{\pi}{2} (k - \beta) \right] \]  \hspace{1cm} (3.24)

\[ V_{be} = \frac{E_h h^3 W_0^2 \pi^2}{96 \lambda a^2} \left[ \kappa_1 \beta^2 + \frac{\kappa_2}{\beta^2} + 2 \alpha_L + 4 \lambda \nu \right] \]  \hspace{1cm} (3.25)

\[ V_{bm} = \frac{E_h h^3 W_0^2 \pi^2}{48 \lambda a^2} (k - \beta) \beta \kappa_1 \]  \hspace{1cm} (3.26)

The parameters \( W_0 \) and \( \beta \) are to be determined from the requirement that

\[ V = V_{el} + V_{be} + V_{bm} \]  \hspace{1cm} (3.27)

shall be a minimum as a function of these quantities. From (3.24), (3.25) and (3.26) the expression for \( V \) can be written in the form:

\[ V = L w_0^2 - M p w_0 \]  \hspace{1cm} (3.28)
where

\[ L = \frac{E_h h^3 \pi^4}{96 \lambda a^2} \left( \frac{\kappa_2}{\tau^3} + \frac{\delta}{\tau} - \kappa_1 \tau + 2 \kappa_1 k \right) \quad (3.29) \]

\[ M = \frac{4a^2}{\pi^2} \left[ \tau + \frac{\pi}{2} (k - \tau) \right] \quad (3.30) \]

\[ \delta = 2 \sigma_{ TL } + 4 \lambda \nu \quad (3.31) \]

The conditions

\[ \frac{dV}{d\omega_0} = 2L \omega_0 - Mp = 0 \quad (3.32) \]

\[ \frac{dV}{d\tau} = \omega_0^2 \frac{dL}{d\tau} - p \omega_0 \frac{dM}{d\tau} = 0 \quad (3.33) \]

lead to the equation

\[ 2L \frac{dM}{d\tau} - M \frac{dL}{d\tau} = 0 \]

for the determination of \( \tau \). After some reduction this equation becomes

\[ \tau^5 - 1.248 k \tau^4 - \frac{3 \delta}{\kappa_1} \tau^3 + 2.752 \frac{\delta}{\kappa_1} k \tau^2 \]

\[ - 5 \frac{\kappa_2}{\kappa_1} \tau + 8.256 \frac{\kappa_2}{\kappa_1} k = 0 \quad (3.34) \]

The parameter \( \tau \) having been found as a root of (3.34) the deflection at the center is found at once by solving (3.32) for \( \omega_0 \)

Then

\[ \omega_0 = \frac{pM}{2L} \]

\[ = \alpha \frac{p\alpha^4}{E_h h^3} \quad (3.35) \]

where

\[ \alpha = \frac{192 \lambda}{\pi^6} \left( \tau + \frac{\pi}{2} (k - \tau) \right) \]

\[ \frac{\kappa_2}{\tau^3} + \frac{\delta}{\tau} - \kappa_1 \tau + 2 \kappa_1 k \]

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Instead of (3.35) we can write

$$\frac{W_0}{h} = \alpha P$$  \hspace{1cm} (3.37)

where

$$P = \frac{pa^4}{E_L h^4}$$  \hspace{1cm} (3.38)

For a plate of isotropic material we can use (3.34), (3.35) and (3.37) if we note that in this case $\kappa_2 = \kappa_4 = 1$, $\lambda = 1 - \sigma^2$ and $\delta = 2$. In the equation for $\lambda$, $\sigma$ denotes Poisson's ratio.
Appendix 4.--Rectangular Plate Under Uniform Load.  
Edges Clamped.  Small Deflections

The conditions satisfied by the deflection at the edges are:

\[ \n = 0, \text{ when } x = 0, x = a, \eta = 0, \eta = \beta = \epsilon b; \quad (4.1) \]

\[ \frac{\partial \n}{\partial x} = 0, \text{ when } x = 0, x = a; \quad (4.2) \]

\[ \frac{\partial \n}{\partial \eta} = 0, \text{ when } \eta = 0, \eta = \beta \quad (4.3) \]

The conditions at the edges \( \eta = 0 \) and \( \eta = \beta \) of the transformed plate correspond to those at the edges \( y = 0 \) and \( y = b \) of the given plate.

The differential equation (2.25) is to be solved subject to the conditions (4.1), (4.2) and (4.3).

Choose as a particular integral of (2.25) satisfying the conditions (4.1)

\[ \n_1 = A x (x-a) \eta (\eta-\beta) \quad (4.4) \]

where

\[ A = p/8D_1 \kappa \quad (4.5) \]

A solution \( \n_2 \) of the homogeneous equation, obtained by setting \( p = 0 \) in (2.25) is to be found such that \( \n = \n_1 + \n_2 \) satisfies all of the conditions (4.1), (4.2) and (4.3). Choose \( \n_2 \) as follows:

\[ \n_2 = A \left\{ \sum_n \frac{a_n}{H_n(\beta)} \left[ \sin \delta_n (\eta-\beta) \sinh \gamma_n \eta \\
+ \sin \delta_n \eta \sinh \gamma_n (\eta-\beta) \right] \sin \lambda_n x \\
+ \sum_m \frac{b_m}{K_m(\alpha)} \left[ \sin c_m (x-a) \sinh d_m x \\
+ \sin c_m x \sinh d_m (x-a) \right] \sin \sigma_m \eta \right\} \quad (4.6) \]
where the summations are extended over positive odd integral values of $\mathcal{M}$ and $\mathcal{N}$ respectively and

$$\lambda_n = n\pi/a, \quad \sigma_m = m\pi/\beta$$

$$\gamma_n = \lambda_n \rho, \quad \delta_n = \lambda_n \sigma, \quad d_m = \sigma_m \rho, \quad c_m = \sigma_m \sigma$$

$$\rho = \sqrt{(1+\kappa)/2}, \quad \sigma = \sqrt{(1-\kappa)/2}$$

$$H_n(\beta) = \gamma_n \sin \delta_n \beta + \delta_n \sinh \gamma_n \beta$$

$$K_m(a) = d_m \sin c_m a + c_m \sinh d_m a$$

The function $\mathcal{W}_2$ satisfies the conditions (4.1). It remains to determine the coefficients $\alpha_n$ and $b_m$ in such a way that the combination

$$\mathcal{W} = \mathcal{W}_1 + \mathcal{W}_2$$

satisfies the remaining conditions (4.2) and (4.3). The procedure is much the same as that for the corresponding problem in the case of the isotropic plate. $^{26,27}$ It is found that $\alpha_n$ and $b_m$ vanish when their subscripts are even and that these letters with odd subscripts satisfy the equations

$$b_m = \frac{8a}{\sigma_m^3} - \frac{8}{\beta} \sum_n \frac{\lambda_n \gamma_n F_n(\beta)}{(\delta_n^2 + \sigma_m^2 + \gamma_n^2)^2 - 4 \delta_n^2 \sigma_m^2} \alpha_n$$

$$\alpha_n = \frac{8 \beta}{a \lambda_n^3} - \frac{8 \lambda_n}{a} \sum_m \frac{\sigma_m d_m G_m(a)}{(c_m^2 + \lambda_n^2 + d_m^2)^2 - 4 c_m^2 \lambda_n^2} b_m$$

where

$$F_n(\beta) = \frac{\cos \delta_n \beta + \cosh \gamma_n \beta}{(\rho/\sigma) \sin \delta_n \beta + \sinh \gamma_n \beta}$$

$$G_m(a) = \frac{\cos c_m a + \cosh d_m a}{(\rho/\sigma) \sin c_m a + \sinh d_m a}$$

$^{26}$ Hencky, H., Darmstadt Dissertation, 1913.

$^{27}$ March, H. W., Trans. American Math. Soc. 27, 307-17, 1925.
From this point on in appendix 4, the numbers \( M \) and \( N \) will be considered to be odd integers.

For purposes of computation it is convenient to write equations (4.12) and (4.13) in the forms

\[
\begin{align*}
b_m &= \frac{B}{m^3} - \sum_n Q_{mn} F_n(\beta) a_n \quad (4.16) \\
\alpha_n &= \frac{C}{n^3} - \sum_m R_{nm} G_m(\alpha) b_m \quad (4.17)
\end{align*}
\]

\(( m = 1, 3, 5 \ldots \); \( n = 1, 3, 5 \ldots \))

where, writing

\[
\frac{b}{a} = k \quad \text{and} \quad s = \varepsilon k = \varepsilon \frac{b}{a} = \rho/a \quad (4.18)
\]

\[
B = 8s^2a^3/\pi^3, \quad C = 8s a^3/\pi^3 \quad (4.19)
\]

\[
Q_{mn} = \frac{8s^2\rho}{\pi} \frac{m \, n^2}{m^4 + 2\kappa m^2 n^2 s^2 + n^4 s^4} \quad (4.20)
\]

\[
R_{nm} = \frac{8s^2\rho}{\pi} \frac{n \, m^2}{m^4 + 2\kappa m^2 n^2 s^2 + n^4 s^4} \quad (4.21)
\]

and \( F_n(\beta) \) and \( G_m(\alpha) \) are defined by (4.14) and (4.15).

Because of the rapid decrease in value of \( Q_n \) and \( b_m \) with increasing \( N \) and \( M \), the first few \( Q_n \)'s and \( b_m \)'s can be found by solving the finite system of equations obtained by replacing the unknowns with higher indices by zero in the first few equations.

\[\text{In D. E. Zilmer's thesis to be presented at the University of Wisconsin, he discusses rigorously the solution of the infinite system of equations for both the orthotropic and the isotropic plate. This was done for the isotropic plate to remedy a defect in the convergence proof of the present author's paper (Trans. Am. Math. Soc. 27, 307-317, 1925). This defect was due to the omission of the factor \( \pi \) in the last term on page 312 of that paper. The method of the convergence proof is not changed essentially. S. Iguchi (See footnotes 1 and 13) was also led to an infinite system of equations. He did not establish the convergence of the process used in solving this system.}\]
The values of \( a_1, a_3, b_1 \) and \( b_3 \) apart from the common factor \( A^3 \) are given in table 41 for plates of various types of plywood and for various ratios of the sides \( b \) and \( a \).

An approximate formula will now be obtained for the deflection at the center of a uniformly loaded plate with clamped edges. As in the case of a plate with simply supported edges a plausible form, depending upon certain parameters, will be assumed for the deflected middle surface. The parameters will then be determined in such a way that the sum of the potential energy of deformation of the plate and the change in the potential energy of the applied load shall be a minimum.

The plate is divided into three portions by the lines \( y = c \) and \( y = b - c \). The following forms are assumed for the deflection in each of these portions:

\[
W = W_0 \sin^2 \frac{\pi x}{a}, \quad 0 < y < b - c;
\]

\[
W = W_0 \sin^2 \frac{\pi x}{a} \sin^2 \frac{\pi y}{2c}, \quad 0 < y < c; \tag{4.22}
\]

and a form corresponding to the latter for the region, \( b - c < y < b \)

Let

\[
c = \frac{\tau a}{2} \tag{4.23}
\]

Then the deflection \( W \) is expressed as a function of the two parameters \( W_0 \) and \( \tau \) which are to be determined in such a way that the total potential energy of the system is a minimum.

Let the symbols \( V_\ell, V_{be} \) and \( V_{bm} \) have the same meaning as in section 3. Using (3.20), (3.21) and (4.22) it is readily found that

\[
V_\ell = - \frac{p W_0 a^2}{4} (2k - \tau) \tag{4.24}
\]

\[
V_{be} = \frac{E_L h^3 \pi^4 W_0^2}{96 \lambda a^2} \left(3 \kappa_1 \tau + \frac{3 \kappa_2}{\tau^3} + \frac{\delta}{\tau}\right) \tag{4.25}
\]

\[
V_{bm} = \frac{E_L h^3 \pi^4 W_0^2 \kappa_1}{12 \lambda a^2} (k - \tau) \tag{4.26}
\]
where $\kappa_1, \kappa_2$ and $\xi$ are defined by (3.22) and (3.31).

Then the total potential energy of the system

$$V = V_L + V_{be} + V_{bm}$$

can be written in the form:

$$V = L \omega_0^2 - M p \omega_0$$  \hspace{1cm} (4.27)

The conditions

$$\frac{\partial V}{\partial \omega_0} = 2 L \omega_0 - M p = 0$$  \hspace{1cm} (4.28)

$$\frac{\partial V}{\partial \tau} = \omega_0^2 \frac{dL}{d\tau} - p \omega_0 \frac{dM}{d\tau} = 0$$  \hspace{1cm} (4.29)

lead to the equation

$$2 L \frac{dM}{d\tau} - M \frac{dL}{d\tau} = 0$$

for the determination of $\tau$. After some reduction this equation becomes

$$\tau^5 - 1.2 k \tau^4 - 0.6 \frac{\delta}{\kappa_1} \tau^3 + \frac{0.4 \delta k}{\kappa_1} \tau^2$$

$$- 3 \frac{\kappa_2}{\kappa_1} \tau + 3.6 \frac{\kappa_2 k}{\kappa_1} = 0$$  \hspace{1cm} (4.30)

The parameter $\tau$ having been found as a root of (4.30), the deflection at the center is found at once by solving (4.28) for $\omega_0$. Then

$$\omega_0 = \alpha \frac{p a^4}{E_l h^3}$$  \hspace{1cm} (4.31)

where

$$\alpha = \frac{12 \lambda}{\pi^4} \frac{2 k - \tau}{3 \frac{\kappa_2}{\kappa_1} + \frac{\delta}{\tau} + 8 \kappa_1 k - 5 \kappa_1 \tau}$$  \hspace{1cm} (4.32)
Instead of (4.31) we can write

\[ \frac{\mathcal{W}_0}{\hbar} = \alpha P \]  

(4.33)

where

\[ P = \frac{p \alpha^4}{E_L h^4} \]  

(4.34)
The edges of the strip, \( x = 0 \) and \( x = a \), figure 30, are simply supported. A concentrated load \( P \) is applied at the point \( x = u, y = 0 \) on the \( x \)-axis.

At first the load\(^{22}\) will be considered to be uniformly distributed along the segment, \( u - \alpha < x < u + \alpha \), of the \( x \)-axis, the intensity of the load per unit length being \( p_0 \). The total load is then \( P = 2p_0\alpha \). Later \( \alpha \) will be made to approach zero. In this case \( p_0 \) will be taken to increase in such a way that the product \( 2p_0\alpha P \) remains constant.

Because of the discontinuity in the loading, it is convenient to determine the deflection and moments in the regions \( y > 0 \) and \( y < 0 \) separately. Consider the region \( y > 0 \). The deflection \( w \) satisfies the differential equation (2.25) with \( p = 0 \) at all points of this region. The boundary conditions are:

\[
\begin{align*}
  w &= 0 \quad \text{when } x = 0, \ x = a, \ \text{and when } y = \infty; \quad (5.1) \\
  \frac{\partial^2 w}{\partial x^2} &= 0 \quad \text{when } x = 0, \ x = a; \quad (5.2) \\
  \frac{\partial w}{\partial y} &= 0 \quad \text{when } y = 0. \quad (5.3)
\end{align*}
\]

A further condition on \( w \) is found from the distribution of vertical shear, \( p_y \), along the line \( y = 0 \). The load on the segment \((u - \alpha, u + \alpha)\) can be represented by the series

\[
\frac{4p_0}{\pi} \sum_{n=1}^{\infty} \frac{\sin \lambda_n u \sin \lambda_n \alpha}{n} \sin \lambda_n x \quad (5.4)
\]

where \( \lambda_n = \frac{n\pi}{a} \). We accordingly require that

\[
\lim_{y \to 0} p_y = -\frac{2p_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \lambda_n u \sin \lambda_n \alpha \sin \lambda_n x \quad (5.5)
\]

\(^{22}\) Nadai, A., Elastische Platten, pp. 78-82 and 85-95. Huber, M. T., Bauingenieur 6, 1925.
From (2.10) using the variable \( \eta = \varepsilon y \) instead of \( y \) we can find the expression for \( p_y \) in terms of \( \psi \). Entering this expression in (5.5) we obtain:

\[
\lim_{\eta \to 0} (-D_2 \varepsilon^3 \frac{\partial^3 \psi}{\partial \eta^3} - K \varepsilon \frac{\partial^3 \psi}{\partial \eta^3}) = -\frac{2p_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \lambda_n u \sin \lambda_n \alpha \sin \lambda_n x \quad (5.6)
\]

By (5.5) and (5.6) it has been arranged that the discontinuity in vertical shear along the \( X \)-axis is given by (5.4). Corresponding to (5.5) the limiting value of \( p_y \) as \( y \) approaches zero from below is equal to the right-hand member of this equation with its sign changed.

The solution of (2.25), with \( p = 0 \), which satisfies (5.1), (5.2), (5.3), and (5.6) is:

\[
\psi = \sum_{n=1}^{\infty} A_n e^{-\lambda_n \eta} (\cos \delta_n \eta + c \sin \delta_n \eta) \sin \lambda_n x \quad (5.7)
\]

where

\[
A_n = \frac{p_0}{\pi \rho D_2 \varepsilon^3} \frac{\sin \lambda_n u \sin \lambda_n \alpha}{n \lambda_n^3} = \frac{p_0 \varepsilon}{\pi \rho D_1} \frac{\sin \lambda_n u \sin \lambda_n \alpha}{n \lambda_n^3} \quad (5.8)
\]

\[c = \rho / \delta = \rho / \sigma.\]

The remaining symbols are defined in the table of notations and in section 3.

We now allow \( \alpha \) to approach zero and \( p_0 \) to increase in such a way that \( 2p_0 \alpha \) remains constant and equal to \( P \).

The coefficient \( A_n \) in (5.7) becomes

\[
A_n = A \frac{\sin \lambda_n u}{\lambda_n^3} \quad (5.9)
\]
\[
A = \frac{P \epsilon}{2 \alpha D_i \rho}
\]  
(5.10)

The bending and twisting moments will be calculated by (2.29).

If we let

\[
\psi = \sum_{n=1}^{\infty} A_n \lambda_n^2 e^{-\kappa \eta} \sin \delta_n \eta \sin \lambda_n \chi
\]  
(5.11)

\[
\chi = \sum_{n=1}^{\infty} A_n \lambda_n^2 e^{-\kappa \eta} \cos \delta_n \eta \sin \lambda_n \chi
\]  
(5.12)

then

\[
\frac{\partial^2 W}{\partial x^2} = -\chi - c \psi
\]

\[
\frac{\partial^2 W}{\partial \eta^2} = -\chi + c \psi
\]

and from (2.29),

\[
m_x = \frac{D_1}{(E_1 E_2)^{1/2}} (\beta \chi - \alpha c \psi)
\]  
(5.13)

\[
m_y = \frac{D_2}{E_2} (\beta \chi + \alpha c \psi)
\]  
(5.14)

where

\[
\alpha = E_L \sigma_{TL} - (E_1 E_2)^{1/2}
\]  
(5.15)
\[ \beta = E_L \sigma_{\pi L} + (E_1 E_2)^{1/2} \] \hfill (5.16)

Expressions for \( \chi \) and \( \psi \) will be obtained in closed form.

Let
\[ \phi = \chi + i \psi \] \hfill (5.17)
\[ = \sum_{n=1}^{\infty} A_n \lambda_n^2 e^{-(\gamma_n - i \delta_n) \eta} \sin \lambda_n x \]
\[ = A \sum_{n=1}^{\infty} e^{-\lambda_n \eta'} \sin \lambda_n x \sin \lambda_n u \] \hfill (5.18)

where
\[ \lambda_n \eta' = (\gamma_n - i \delta_n) \eta = \lambda_n (\rho - i \sigma) \eta \] \hfill (5.19)

From (5.18) we obtain
\[ \phi = -\frac{A \alpha}{2\pi} \Re \sum_{n=1}^{\infty} \left[ e^{-n(\xi + i \zeta)} - e^{-n(\xi + i \zeta')} \right] \] \hfill (5.20)

where
\[ \xi = \frac{\pi \eta'}{\alpha}, \quad \zeta = \frac{\pi}{\alpha} (x + u), \quad \zeta' = \frac{\pi}{\alpha} (x - u) \]

and \( \Re \) denotes the real part of the expression following it.

Now, setting
\[ e^{-\xi + i \zeta} = z \]
\[ \mathcal{R} \sum \frac{\phi}{\pi} = \mathcal{R} \sum \frac{z^n}{n} = -\mathcal{R} \log (1 - z) = -\frac{1}{2} \log \left[ z e^{-\xi} (\cosh \xi - \cos \xi) \right] \]

Treating the second term in (5.20) in the same way

\[ \phi = -\frac{Aa}{4\pi} \log \frac{\cosh \xi - \cos \xi'}{\cosh \xi - \cos \xi} \]

In the notation of (5.19)

\[ \phi = -\frac{Aa}{4\pi} \log \frac{\cosh \frac{\pi \eta'}{\alpha} - \cos \frac{\pi (x-u)}{\alpha}}{\cosh \frac{\pi \eta}{\alpha} - \cos \frac{\pi (x+u)}{\alpha}} \]  \hspace{1cm} (5.21)

Recalling (5.17), \( \chi \) and \( \psi \) can be found from the relations

\[ \chi = \mathcal{R} \phi, \quad \psi = \mathcal{I} \phi \]

where \( \mathcal{R} \) and \( \mathcal{I} \) denote the real and imaginary parts, respectively, of the expressions following them. It follows from (5.19) and (5.21) that:

\[ \chi = -\frac{Aa}{4\pi} \log \left( \frac{H^2 + B^2}{C^2 + B^2} \right)^{\frac{1}{2}} \]  \hspace{1cm} (5.22)

\[ \psi = \frac{Aa}{4\pi} \tan^{-1} \frac{B(C-H)}{CH + B^2} \]  \hspace{1cm} (5.23)

where

\[ H = \cosh \frac{\pi \rho \eta}{\alpha} \cos \frac{\pi \sigma \eta}{\alpha} - \cos \frac{\pi}{\alpha} (x-u) \]  \hspace{1cm} (5.24)

\[ B = \sinh \frac{\pi \rho \eta}{\alpha} \sin \frac{\pi \sigma \eta}{\alpha} \]  \hspace{1cm} (5.25)
\[ C = \cosh \frac{\pi \rho \gamma}{\alpha} \cos \frac{\pi \sigma \gamma}{\alpha} - \cos \frac{\pi}{\alpha} (x + u) \quad (5.26) \]

In the vicinity of the point of loading \((u,0)\) the following approximate expressions for \(\chi\) and \(\psi\) are readily obtained:

\[ \chi = -\frac{Aa}{4\pi} \log \frac{\kappa^2 [\eta^4 + 2\kappa(x-u)^2 \eta^2 + (x-u)^4]^{1/2}}{2a^2 (1 - \cos \frac{2\pi u}{\alpha})} \quad (5.27) \]

\[ \psi = \frac{Aa}{4\pi} \tan^{-1} \frac{2\rho \sigma \eta^2}{\kappa \eta^2 + (x-u)^2} \quad (5.28) \]

It is to be observed that \(\chi\) becomes infinite at the point of loading and that the limiting value of \(\psi\) depends upon the path of approach to this point.

The deflection at any point in the strip can be calculated from equation (5.7) with the coefficients given by (5.9) as the series is rapidly convergent. The bending moments \(M_x\) and \(M_y\) can be calculated by the use of (5.13) and (5.14), the functions \(\chi\) and \(\psi\) that are needed, being found from (5.22) and (5.23).

For isotropic material \(\rho = 1\) and \(\sigma = 0\) in this case the function \(\chi\) as given by (5.23) is identical, apart from a constant factor, with the function \(\phi\) which was obtained by Nadai\(^{30}\) in another way. The function \(\psi\) reduces to zero. But the product \(C \psi\) which occurs in the expressions (5.13) and (5.14) for the moments, approaches the limit \(-\psi \frac{\partial \chi}{\partial y}\) as \(\kappa\) approaches 1. The expressions for the moments \(M_x\) and \(M_y\) accordingly reduce to the known expressions for the isotropic case.

\(^{30}\) Nadai, A., Elastische Platten, p. 95.
Appendix 6.--Infinite Strip (Long Narrow Rectangular Plate). Uniform Load Applied over a Small Area. Edges Simply Supported

A uniform load $q$ per unit area acts over a rectangular area $2\alpha \times 2\gamma$ whose center is at $(U, 0)$ as shown in figure 21.

In appendix 5 the deflection associated with a line load $p_0$ per unit length of the segment of the $x$-axis between $x = U - \alpha$ and $x = U + \alpha$ was found to be given by (5.7) for points in the upper half of the infinite strip. The loaded line segment will now be replaced by a uniformly loaded strip of width $d\gamma$. The corresponding deflection at points for which $\gamma$ (or $\eta$) is positive is again given by (5.7) if the coefficients $A_n$ (see (5.8)) are modified by replacing $p_0$ by $qdy$ where $q$ is the uniform load per unit area. It will be convenient to use the variable $\eta$ instead of $\gamma$ and write

$$p_0 = qdy = \frac{q}{\varepsilon} \ d\eta \quad (6.1)$$

The properly modified forms of (5.7) and (5.8) are then

$$\mathcal{W} = \sum_{n=1}^{\infty} B_n d\eta e^{-h\eta} (\cos \delta_n \eta + c \sin \delta_n \eta) \sin \lambda_n x \quad (6.2)$$

where

$$B_n = \frac{q}{\pi D_1 \varepsilon} \frac{\sin \lambda_n u \sin \lambda_n \alpha}{n \lambda_n^3} \quad (6.3)$$

In writing (6.3) we have used the relation $D_2 \varepsilon^4 = D_1$.

The coefficient $C_n$ in (6.2), has the same meaning as in (5.7).

As the transformation, $\eta = \varepsilon \gamma$, is being used, the dimensions of the loaded area on the transformed plate are $2\alpha \times 2\gamma$ where

$$\gamma = \varepsilon \gamma \quad (6.4)$$

The boundaries of the loaded area are the lines $x = U - \alpha$, $x = U + \alpha$, $\gamma = -\tau$, and $\eta = \gamma$.

To find the deflection due to the loaded rectangular area consider a horizontal strip of width $d\gamma$ at $\gamma = \gamma$ within the

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boundaries of the loaded area on the transformed plate and calculate the deflection at the point \((x, \eta)\) due to the load on this strip. If \(\eta > \nu\), this deflection is given by (6.2) with \(\eta\) replaced by \(\eta - \nu\) and \(d\eta\) by \(d\nu\). If \(\eta < \nu\) these replacements are to be made in the equation corresponding to (6.2) for \(\eta < 0\).

For points for which \(\eta > \tau\) (namely, \(\nu > \tau\)) the deflection is obtained by integrating between the limits \(-\tau\) and \(\tau\), the expression for the deflection due to the strip of width \(d\nu\) at \(\eta = \nu\). The result is

\[
\varphi = -\sum_{n=1}^{\infty} \frac{B_n \sin \lambda_n x}{\delta_n} \left[ e^{\kappa(\tau-\eta)} \sin[\delta_n (\tau - \eta) - \theta] + e^{-\kappa(\tau+\eta)} \sin[\delta_n (\tau + \eta) + \theta] \right] \tag{6.5}
\]

where

\[
\cos \theta = \kappa, \quad \sin \theta = \sqrt{1 - \kappa^2} \tag{6.6}
\]

For the region \(\eta < -\tau\), \(\eta\) in (6.5) is to be replaced by \(-\eta\) throughout.

If \(-\tau < \eta < \tau\), the deflection is found as the sum of two integrals, one between the limits \(-\tau\) and \(\eta\), the other between the limits \(\eta\) and \(\tau\). The result is

\[
\varphi = \sum_{n=1}^{\infty} \frac{B_n \sin \lambda_n x}{\delta_n} \left[ 2\sin \theta - e^{-\kappa(\tau+\eta)} \sin[\delta_n (\tau + \eta) + \theta] + e^{-\kappa(\tau-\eta)} \sin[\delta_n (\tau - \eta) - \theta] \right] \tag{6.7}
\]

The moments can be calculated from (6.5) and (6.7) with the aid of (2.29).
Appendix 7.—Rectangular Plate. Load Concentrated at a Point or Applied Over a Small Area. Edges Simply Supported

The method of a suitable distribution of positive and negative loads as described in section 7 can be applied for both types of loads. However, for the case of a point load acting at a point on the central line \( y = 0 \) (see figure 31) the method used by Timoshenko,\(^{31}\) for the corresponding type of loading of an isotropic plate leads more directly to the result.

With the choice of axes shown in figure 31, a concentrated point load is applied at the point \((C, 0)\). Take the following solution of the equation (2.25) with \( p = 0 \), for the upper half of the plate:

\[
W = \sum_{n=1}^{\infty} Y_n \sin \lambda_n x
\]

(7.1)

where

\[
Y_n = \alpha_n \sinh \gamma_n \eta \sin \delta_n \eta + b_n \sinh \gamma_n \eta \cos \delta_n \eta \\
+ c_n \cosh \gamma_n \eta \sin \delta_n \eta + d_n \cosh \gamma_n \eta \cos \delta_n \eta
\]

(7.2)

The conditions on the edges \( x = 0 \) and \( x = a \) are satisfied by (7.1). The coefficients in (7.2) are to be determined so that

\[
\frac{1}{\gamma} \frac{\partial W}{\partial \gamma} = 0 \text{ on } y = 0, \text{ that the condition for discontinuity in vertical shear along } y = 0 \text{ is satisfied (see section 5 and appendix 5) and that } W = 0 \text{ and } \frac{\partial^2 W}{\partial \gamma^2} = 0 \text{ on } \eta = \frac{\beta}{2} = \frac{\epsilon_b}{2}.
\]

It is found that

\[
\alpha_n = -\frac{P \epsilon}{2 a D_i \rho \sigma} \frac{\sigma \sin \delta_n \beta + \rho \sinh \gamma_n \beta}{\cos \delta_n \beta + \cosh \gamma_n \beta} \frac{\sin \lambda_n c}{\lambda^3_n}
\]

\[
b_n = -\frac{P \epsilon}{2 a D_i \rho} \frac{\sin \lambda_n c}{\lambda^3_n}
\]

\[
c_n = -\frac{\rho}{\sigma} b_n
\]

\[
d_n = -\frac{P \epsilon}{2 a D_i \rho \sigma} \frac{\rho \sin \delta_n \beta - \sigma \sinh \gamma_n \beta}{\cos \delta_n \beta + \cosh \gamma_n \beta} \frac{\sin \lambda_n c}{\lambda^3_n}
\]

\(^{31}\)Timoshenko, S., Bauingenieur 3, 51, 1922.
With these values of the coefficients the deflection at the center \( x = a/2 \), \( \eta = 0 \) due to a load at this point is found to be given by the formula

\[
W_0 = \frac{Pa^2}{E_i h^3} \frac{6\lambda \epsilon}{\rho \pi^3} \sum_n \frac{1}{n^3} \frac{\sinh \kappa_n \beta - \rho/\sigma \sin\delta_n \beta}{\cosh \kappa_n \beta + \cos \delta_n \beta}
\]  

(7.3)

\( n = 1, 3, 5 \ldots \)

In obtaining (7.3) the distance \( C \) in the expressions for the coefficients \( a_n, b_n \), etc., has been set equal to \( a/2 \). The symbols which occur in (7.3) are defined in the table of notations.

For an infinite strip, \( b \to \infty \), it is found from (7.3) that the deflection \( W_{0\infty} \) at the center due to a point load at the center of the strip is given by the equation:

\[
W_{0\infty} = \frac{Pa^2}{E_i h^3} \frac{6\lambda \epsilon}{\rho \pi^3} \left(1 + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \cdots\right)
\]

\[
= 1.051 \frac{Pa^2}{E_i h^3} \frac{6\lambda \epsilon}{\rho \pi^3},
\]  

(7.4)

a result that is in agreement with one which could be obtained from (5.7), (5.8), (5.9), and (5.10) of appendix 5.

The factor \( \gamma \) in the equation

\[
W_0 = \gamma W_{0\infty}
\]  

(7.5)

is to be calculated from the formula

\[
\gamma = \frac{1}{1.051} \sum_n \frac{1}{n^3} \frac{\sinh \kappa_n \beta - (\rho/\sigma) \sin \delta_n \beta}{\cosh \kappa_n \beta + \cos \delta_n \beta}
\]  

(7.6)

\( n = 1, 3, 5, 7 \ldots \)
Appendix 8.—The Differential Equations for the Deflection of a Plywood Plate. Large Deflections

When the deflections of a plate become so large that direct stresses are developed in addition to the usual bending stresses the state of strain in the plate is a superposition of two states of strain, one associated with the bending stresses and a second associated with the direct stresses. The components of the former, given by equations (2.2) vary linearly across the thickness of the plate. The components of the latter are constant across the thickness of the plate. They are given by the equations\(^{32}\)

\[
e'_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2
\]

\[
e'_{yy} = \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2
\]

\[
e'_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y}
\]

(8.1)

where \(u, v,\) and \(w\) denote the components of the displacement of a point in the middle surface of the plate.

The corresponding direct stresses at a point in any one of the plies are given by (see (2.3))

\[
X'_{x} = \frac{E_x}{\lambda} (e'_{xx} + \sigma_{yx} e'_{yy})
\]

\[
Y'_{y} = \frac{E_y}{\lambda} (e'_{yy} + \sigma_{xy} e'_{xx})
\]

\[
X'_{y} = \mu_{xy}\ e'_{xy}
\]

(8.2)

where \(E_x, E_y\) and \(\mu_{xy}\) denote the values of these constants in the ply under consideration and \(\lambda = 1 - \sigma_{xy} \sigma_{yx}\). The mean stress components \(X'_{x}, Y'_{y},\) and \(X'_{y}\) are obtained by averaging the stress components \(X'_{x}, Y'_{y},\) and \(X'_{y}\) over the thickness of the plate.

It follows readily that, for plywood with flat-grained plies,

\[ \bar{X}_x' = \frac{1}{\lambda} \left( E_a e'_{xx} + E_L \sigma_{TL} e'_{yy} \right) \]
\[ \bar{Y}_y' = \frac{1}{\lambda} \left( E_b e'_{yy} + E_L \sigma_{TL} e'_{xx} \right) \]
\[ \bar{X}_y' = \mu_{LT} e'_{xy} \tag{8.3} \]

where

\[ E_a = \frac{1}{h} \int_{-h/2}^{h/2} E_x \, dz \]
\[ E_b = \frac{1}{h} \int_{-h/2}^{h/2} E_y \, dz, \quad \lambda = 1 - \sigma_{LT} \sigma_{TL} \tag{8.4} \]

The quantities \( E_a \) and \( E_b \) may be called the "mean moduli in stretching" in the \( X \) and \( Y \) directions, respectively.

Denote the mean forces, resulting from the mean stress components, per unit length of edge of an element of the plate such as that shown in figure 29, by \( n_x, n_y, \) and \( n_{xy} \). Then

\[ n_x = h \bar{X}_x', \quad n_y = h \bar{Y}_y', \quad n_{xy} = h \bar{X}_y' \tag{8.5} \]

Since the deflection is assumed to be small in comparison with the length and breadth of the plate the conditions for equilibrium of the forces \( n_x, n_y, \) and \( n_{xy} \) or of the stress components \( \bar{X}_x', \bar{Y}_y', \) and \( \bar{X}_y' \)

---

33 Price, A. T., Phil. Trans. A228, 1-62, 1928. Apparent Young's modulus for stretching, p. 41. The definition of this modulus as here given differs from that given by Price by a term whose value is small. See his equation (13.72) and his discussion of plywood on pages 50-52.
will be the same as if the plate were plane and in equilibrium under forces acting in its plane. These conditions are

\[
\frac{\partial \overline{X}_x'}{\partial x} + \frac{\partial \overline{X}_y'}{\partial y} = 0
\]

\[
\frac{\partial \overline{Y}_y'}{\partial x} + \frac{\partial \overline{Y}_y'}{\partial y} = 0
\]

(8.6)

Accordingly, there exists a stress function \( F \) such that

\[
\overline{X}_x' = \frac{\partial^2 F}{\partial y^2}, \quad \overline{Y}_y' = \frac{\partial^2 F}{\partial x^2}, \quad \overline{X}_y' = -\frac{\partial^2 F}{\partial x \partial y}
\]

(8.7)

The elimination of \( U \) and \( V \) from the system of equations (8.1) leads to the following relation connecting the components of strain:

\[
\frac{\partial^2 \varepsilon_{xx}'}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}'}{\partial x^2} - \frac{\partial^2 \varepsilon_{xy}'}{\partial x \partial y} = \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2}
\]

(8.8)

This equation replaces one of the conditions of compatibility of the strain components.

From (8.3) and (8.7)

\[
\begin{align*}
\varepsilon_{xx}' &= \frac{E_b}{H} \frac{\partial^2 F}{\partial y^2} - \frac{E_L \sigma_{TL}}{H} \frac{\partial^2 F}{\partial x^2} \\
\varepsilon_{yy}' &= \frac{E_b}{H} \frac{\partial^2 F}{\partial x^2} - \frac{E_L \sigma_{TL}}{H} \frac{\partial^2 F}{\partial y^2} \\
\varepsilon_{xy}' &= -\frac{1}{\mu_{LT}} \frac{\partial^2 F}{\partial x \partial y}
\end{align*}
\]

(8.9)
\[ H = \frac{E_a E_b - E_L^2 \sigma_{tt}}{\lambda} \quad (8.10) \]

The substitution of (8.9) in (8.8) leads to the following differential equation:

\[ \frac{E_a}{H} \frac{\partial^4 F}{\partial x^4} + \left( \frac{1}{\mu_{LT}} - \frac{2E_L \sigma_{tt}}{H} \right) \frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{E_b}{H} \frac{\partial^4 F}{\partial y^4} \]
\[ = \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \quad (8.11) \]

The condition for the equilibrium of an element of the plate under the vertical components of the forces acting on it is expressed by the equation:

\[ \frac{\partial p_x}{\partial x} + \frac{\partial p_y}{\partial y} + n_x \frac{\partial^2 w}{\partial x^2} + 2n_{xy} \frac{\partial^2 w}{\partial x \partial y} + n_y \frac{\partial^2 w}{\partial y^2} + p = 0 \quad (8.12) \]

where \( p_x \) and \( p_y \) are defined by (2.9).

Now (See (2.11) and (2.12))

\[ \frac{\partial p_x}{\partial x} + \frac{\partial p_y}{\partial y} = -\left( D_1 \frac{\partial^4 w}{\partial x^4} + 2K \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) \quad (8.13) \]

Using (8.5), (8.7) and (8.13), equation (8.12) becomes

\[ D_1 \frac{\partial^4 w}{\partial x^4} + 2K \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 w}{\partial y^4} \]
\[ = p + h \left[ \frac{\partial^2 F}{\partial y^2} \frac{\partial^2 w}{\partial x^2} - 2 \frac{\partial^2 F}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 F}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right] \quad (8.14) \]

Equations (8.11) and (8.14) constitute a pair of simultaneous equations from whose solution under appropriate boundary conditions the deflection and stresses of a given plate under a given load are to be determined.
Appendix 9.—Infinite Strip (Long Narrow Plate). Large Deflections. Uniformly Distributed Load. Edges Simply Supported

Consider an infinite strip of plywood of width \( \alpha \). Let the uniformly distributed load per unit area be denoted by \( \rho \). The edges \( x = 0 \) and \( x = \alpha \) are taken to be simply supported and restrained from movement in a direction perpendicular to the length of the plate. Under the assumed uniform loading the deflection \( W \) will be independent of \( y \). The component \( V \) of the displacement (the component parallel to the \( Y \)-axis) of points in the middle surface will vanish. Further, the component \( U \) of the displacement (the component parallel to the \( X \)-axis) of such points will be independent of \( y \). Consequently

\[
\frac{\partial U}{\partial y} = 0 \tag{9.1}
\]

It follows from (8.1) and (9.1), since \( V \) vanishes and \( U \) and \( W \) are independent of \( y \), that

\[
e_{yy}' = 0, \quad e_{xy}' = 0 \tag{9.2}
\]

Then from (8.3) the mean components of the direct stress system are:

\[
\bar{X}_x' = \frac{E\alpha}{\lambda} e_{xx}'
\]
\[
\bar{Y}_y' = \frac{E\tau L}{\lambda} e_{xx}' \tag{9.3}
\]
\[
X_y' = 0
\]

From the equations of equilibrium (8.6), since \( \bar{X}_y' = 0 \), it follows that \( \bar{X}_x' \) is a function of \( y \) alone. But this function must reduce to a constant since, from the type of loading, it is clear that all components of stress and strain are independent of \( y \). Hence

\[
\bar{X}_x' = g \tag{9.4}
\]

where \( g \) is a constant.
It follows from the first of equations (9.3) that \( e'_{xx} \)
is constant and hence from the second of (9.3) that \( \bar{V}_y' \) is constant.

Since the deflection \( W \) is independent of \( y \) the differential equation (8.14) becomes

\[
D, \frac{d^4W}{dx^4} = p + gh \frac{d^2W}{dx^2}
\]

(9.5)

In writing this equation \( \frac{\partial^2 F}{\partial y^2} = \bar{X}_x' \)

has been replaced by the constant \( g \) in accordance with (9.4). Because of the simplicity of the stress system all the information that could be obtained from the differential equation (8.11) is already contained in (9.4) combined with (9.3). It is to be observed that \( g \)
is constant for a given load \( p \) but that it depends upon \( p \). The quantity \( g \) therefore enters the solution as a parameter.

Equation (9.5) can be written

\[
\frac{d^4W}{dx^4} - k^2 \frac{d^2W}{dx^2} = \frac{p}{D,}
\]

(9.6)

where

\[
k^2 = gh/D,
\]

(9.7)

The solution of (9.6) is

\[
W = c_1x + c_2 + A \sinh kx + B \cosh kx - \frac{px^2}{2k^2D,}
\]

(9.8)

On determining the constants in (9.8) to satisfy the conditions on
the simply supported edges, viz.,

\[
W = 0, \quad \frac{d^2W}{dx^2} = 0 \quad \text{when } x = 0 \quad \text{and } \quad x = a
\]

it is found that

\[
W = \frac{p}{2k^2D,} \left[ \frac{2}{k^2} \cosh k(x-\frac{a}{2}) - \cosh \frac{ka}{2} + x(a-x) \right]
\]

(9.9)
With the aid of this expression it is possible to obtain a relation connecting \( p \) and \( q \) or \( p \) and \( k \), since \( q \) and \( k \) are connected by (9.7). From (9.3)

\[
e'_{xx} = \frac{\lambda}{E_a} \chi' = \frac{\lambda}{E_a} q
\]

Then

\[
\int_{0}^{a} e'_{xx} \, dx = \frac{\lambda a}{E_a} q
\]

(9.10)

Further from (8.1)

\[
e'_{xx} = \frac{du}{dx} + \frac{1}{2} \left( \frac{dw}{dx} \right)^2
\]

Hence

\[
\int_{0}^{a} e'_{xx} \, dx = \frac{1}{2} \int_{0}^{a} \left( \frac{dw}{dx} \right)^2 \, dx
\]

(9.11)

since, under the assumed conditions at the edges, the displacement \( U \) vanishes when \( x = 0 \) and when \( x = a \).

On equating the right hand members of (9.10) and (9.11), calculating \( \frac{dw}{dx} \) from (9.9) and performing the integration it is found that

\[
\frac{\lambda a q}{E_a} = \frac{p^2}{2 k D_i} \left[ 5 \tan h \frac{ka}{2} - 5 \frac{ka}{2} \right.
\]

\[
+ \frac{ka}{2} \tan h^2 \frac{ka}{2} + \frac{2}{3} \left( \frac{ka}{2} \right)^3 \right]
\]

(9.12)

The quantity \( q \) is to be expressed in terms of \( k \) and \( D_i \) with the aid of (9.7) and \( D_i \) in terms of \( E_i \) and \( h \) with the aid of (2.14). The resulting equation can then be solved for the quantity,

\[
\rho = \frac{p a^4}{E_i h^4}
\]

(9.13)
in terms of the quantity

\[ \frac{k \alpha \eta}{2} = \eta \]  \hspace{1cm} (9.14)

It is found that

\[ P = \frac{4}{3 \lambda \sqrt{3}} \left( \frac{E_I}{E_A} \right)^{\frac{1}{2}} \frac{E_I}{E_L} \kappa(\eta) \]  \hspace{1cm} (9.15)

where

\[ \kappa(\eta) = \frac{\eta^4}{\left[ (5 \tanh \eta)/\eta - 5 + \tanh^2 \eta + 2 \eta^2/3 \right]^{\frac{1}{2}}} \]  \hspace{1cm} (9.16)

From (9.9) the deflection on the central line \( x = a/2 \) is given by

\[ w_0 = \frac{p}{2k^2D} \left[ \frac{2}{k^2} \left( 1 - \cosh \frac{k \alpha}{2} \right) + \frac{a^2}{4} \right] \]

Using the abbreviations (9.13) and (9.14) it follows readily that

\[ \frac{w_0}{h} = \frac{3P\lambda E_I}{8E_I} \left( 1 - \frac{2}{\eta^2} \frac{\cosh \eta - 1}{\cosh \eta} \right) \]  \hspace{1cm} (9.17)

From (9.7) the following expression for \( g \), the mean direct stress over the thickness of the plate, is obtained:

\[ g = \frac{E_I}{3\lambda} \frac{h^2}{a^2} \eta^2 \]  \hspace{1cm} (9.18)

The bending stress (tension or compression) at a point in the plane whose coordinate with respect to the middle plane is \( Z \) is given by

\[ X_x = \frac{E_x}{\lambda} e_{xx} = -\frac{E_x}{\lambda} \frac{d^2w}{dx^2} \]  \hspace{1cm} (9.19)
where \( E_x \) denotes the value of \( E \) in a direction parallel to the x-axis in the plane in question. Then at a point on the surface of the plate, \( z = h/2 \),

\[
X_x = \frac{E_x h}{2 \lambda} \frac{\rho}{k^2 D_i} \left( 1 - \cosh k \left( x - \frac{a}{2} \right) \right) \cosh \frac{ka}{2} \]

This stress in a face ply is clearly a maximum along the central line, \( x = a/2 \), of the plate. Denoting this maximum bending stress in a face ply by \( S \) it follows that

\[
S = \frac{3}{2} \frac{P E_x E_L}{E_i} \frac{h^2}{a^2} \Theta(\eta) \tag{9.20}
\]

where

\[
\Theta(\eta) = \frac{1}{\eta^2} \frac{\cosh \eta - 1}{\cosh \eta} \tag{9.21}
\]

and \( P \) and \( \eta \) are defined by (9.13) and (9.14).

Using (9.21) the relation (9.17) can be written

\[
\frac{w_q}{h} = \frac{3P\lambda E_L}{8E_i} \psi(\eta) \tag{9.22}
\]

where

\[
\psi(\eta) = \frac{1}{\eta^2} \left[ 1 - 2 \Theta(\eta) \right] \tag{9.23}
\]

The central deflection, the mean direct stress and the maximum bending stress in a face ply that are associated with a given load are expressed by equations (9.22), (9.16), and (9.20), respectively, in terms of the parameter \( \eta \). The value of this parameter corresponding to a given load can be determined from equation (9.15). Values of the function \( \psi(\eta) \) appearing in this equation are given in table 42. The quantity \[ \frac{3\lambda P E_L}{4E_i} \left( \frac{3E_o}{E_i} \right)^{1/2} \] is to be calculated for the given load.
using the definition (9.13) of $P$. This is the value of the function $K(\eta)$ associated with the given load. The corresponding value of the parameter $\eta$ is to be found from table 42 or from a curve constructed from this table. The parameter $\eta$ having been determined, the values of the central deflection, the mean direct stress and the maximum bending stress in a face ply can be calculated with the aid of equations (9.22), (9.18), and (9.20), respectively. The values of the functions $\Theta(\eta)$ and $\Psi(\eta)$ that are needed in these calculations are given in table 42.

**Approximate formulas.**--It is possible to replace the exact formulas just obtained by very accurate approximate formulas connecting the load and the stresses with the deflection at the center. With the aid of these formulas the calculations involved in any given case are greatly simplified.

From equations (9.15) and (9.22) we obtain

$$\frac{\omega_0}{h} = \eta F(\eta) \left(\frac{E_1}{E_a}\right)^{\frac{1}{2}}$$

where

$$F(\eta) = \frac{K(\eta) \psi(\eta)}{2 \eta \sqrt{3}}$$

With the aid of table 42 it is found that $F(\eta)$ is nearly constant and that it may be replaced by $F = 0.360$, the maximum error being less than 2 percent for the range of values of $\eta$ in which we are interested. Hence the following linear relation holds approximately between $\omega_0/h$ and $\eta$:

$$\frac{\omega_0}{h} = 0.360 \left(\frac{E_1}{E_a}\right)^{\frac{1}{2}} \eta$$

(9.24)

From (9.18) and (9.24) it follows that

$$g = \frac{2.572 E_a}{\lambda} \left(\frac{h}{a}\right)^2 \left(\frac{\omega_0}{h}\right)^2$$

(9.25)

---

This fact is shown by the following table of values:

<table>
<thead>
<tr>
<th>$\eta$</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F(\eta)$</td>
<td>0.3661</td>
<td>0.3642</td>
<td>0.3612</td>
<td>0.3585</td>
<td>0.3568</td>
<td>0.3559</td>
</tr>
<tr>
<td>$\eta$</td>
<td>12</td>
<td>14</td>
<td>16</td>
<td>18</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>$F(\eta)$</td>
<td>0.3553</td>
<td>0.3549</td>
<td>0.3546</td>
<td>0.3544</td>
<td>0.3544</td>
<td></td>
</tr>
</tbody>
</table>

-75-
For spruce $\lambda$ may be taken to be 0.99. For other species this value may probably be used with slight error.

An approximate relation connecting $P$ and $\frac{\nu_0}{h}$ can be obtained from (9.17). For large values of $\eta$ we have approximately

$$\frac{\nu_0}{h} = \frac{3P\lambda E_L}{8E_i} \frac{1}{\eta^2} \left(1 - \frac{2}{\eta^2}\right)$$

Solving this equation for $P$ we have, to the same degree of approximation,

$$P = \frac{16E_i}{3\lambda E_L} \frac{\nu_0}{h} + \frac{8E_i}{3\lambda E_L} \eta^2 \frac{\nu_0}{h}$$

On substituting for $\eta$ its expression in terms of $\frac{\nu_0}{h}$ from (9.24)

$$P = \frac{16E_i}{3\lambda E_L} \frac{\nu_0}{h} + \frac{8E_a}{3\lambda F^2 E_L} \left(\frac{\nu_0}{h}\right)^3 \tag{9.26}$$

where $F = 3.60$

Equation (9.26) is approximately correct for large values of $\eta$ and hence by (9.24) for large values of the deflection.

For small values of $\eta$ we obtain from (9.17), using the Maclaurin's series for $\cosh \eta$ and equation (9.24), the approximate relation

$$P = \frac{32E_i}{5\lambda E_L} \frac{\nu_0}{h} + \frac{2.602}{\lambda F^2} \frac{E_a}{E_L} \left(\frac{\nu_0}{h}\right)^3 \tag{9.27}$$

On comparing (9.26) and (9.27) it is seen that the second terms agree to within about 2.5 percent while the first terms differ to a greater extent. Since the first term is important in comparison with the second when $\frac{\nu_0}{h}$ is small the value of this term for small values of $\frac{\nu_0}{h}$ (or of $\eta$) as it is found in (9.27) is to be used in setting up an empirical formula, especially since this term becomes
of diminishing relative importance with increasing $\frac{W_0}{h}$. Similar reasoning leads to the use of the second term as found in (9.26) corresponding to large values of $\frac{W_0}{h}$ (or of $\eta$). Hence we write

$$P = A \frac{W_0}{h} + B \left(\frac{W_0}{h}\right)^3$$

(9.28)

where

$$A = \frac{6.4 E_i}{\lambda E_L}$$

(9.29)

$$B = \frac{20.6 E_0}{\lambda E_L}$$

(9.30)

It will be noted that the first term of (9.28) expresses the result obtained from the usual theory of thin plates when the deflections are assumed to be small.

An empirical formula for the maximum bending stress in the face plies can be set up with the aid of (9.20) and (9.22). These equations lead to

$$S = \alpha \frac{E_x}{\lambda} \left(\frac{h}{a}\right)^2 \frac{W_0}{h}$$

(9.31)

where

$$\alpha = 4 \Theta(\eta) / \psi(\eta)$$

(9.31a)

and $E_x$ denotes the value of $E$ in a face ply in a direction perpendicular to the edge of the plate. It is found that $\alpha$ ranges from the value 4.8 for small $\eta$ (or $W_0/h$) to 4.0 for large $\eta$.

By using the intermediate value $\alpha = 4.4$, the bending stresses associated with small deflections will be underestimated while those associated with large deflections will be overestimated. The resulting percentage error in total stress, that is, direct stress plus maximum bending stress, will usually be small. Accordingly we write

$$S = 4.4 \frac{E_x}{\lambda} \left(\frac{h}{a}\right)^2 \frac{W_0}{h}$$

(9.32)
A more nearly exact value of the factor $\alpha$ which is taken to be 4.4 in formula (9.32) can be obtained from the curve of figure 27 of section 9 with the aid of (9.24). The latter procedure is recommended.

Equation (9.32) is an approximate expression for the maximum bending stress in a face ply. The stress in an adjacent ply is to be calculated by the formulas to be given below.

The approximate formulas (9.25), (9.28), and (9.31) can be used for isotropic plates. When so used all letters $E$ with subscripts are to be replaced by $E_1$ and $\lambda$ by $1 - \sigma^2$ where $\sigma$ is the Poisson's ratio.

Formulas will now be given for calculating the direct stress in each ply in a given plate from the mean direct stress as found by (9.18) or (9.25). It is to be recalled that it has been assumed that the plies are of equal thickness.

**Three-Ply Plate**

Let

\[ g_1 \text{ = direct stress in a face ply} \]
\[ g_2 \text{ = direct stress in a center ply} \]
\[ g \text{ = mean direct stress} \]
\[ E_x \text{ = the value of } E \text{ in a face ply in the direction parallel to the } X\text{-axis} \]
\[ E_y \text{ = the value of } E \text{ in a face ply in the direction parallel to the } Y\text{-axis} \]
\[ r \text{ = } \frac{E_y}{E_x} \]

The ratio $r$ will be $E_T/E_L$ for plates of type 3X and $E_L/E_T$ for plates of type 3Y. Since $\epsilon_{xx}'$ is the same for all plies and all other strain components are zero, it follows readily from (8.2), (9.2), and (9.3) that

\[ \frac{g_2}{g_1} = r \]
Now
\[ \frac{2g_1 + g_2}{3} = g \]

Hence
\[ g_1 = \frac{3}{2 + r} \quad g \]  \hspace{1cm} (9.33)
\[ g_2 = \frac{3r}{2 + r} \quad g \]  \hspace{1cm} (9.33a)

**Five-Ply Plate**

Let
\[ g_1 = \text{direct stress in the face and center plies} \]
\[ g_2 = \text{direct stress in the intermediate plies} \]

\( E_X \) and \( E_Y \) refer to the values of \( E \) in a face ply and \( r = E_Y/E_X \).

Hence, \( r = E_T/E_L \) in plates of type 5X and \( r = E_L/E_T \) in plates of type 5Y. Then
\[ \frac{g_2}{g_1} = r \]
\[ \frac{3g_1 + 2g_2}{5} = g \]

Hence
\[ g_1 = \frac{5}{3 + 2r} \quad g \]  \hspace{1cm} (9.34)
\[ g_2 = \frac{5r}{3 + 2r} \quad g \]  \hspace{1cm} (9.35)

Formulas will now be obtained for finding the maximum bending stress in a given ply in a given plate from the maximum bending stress in a face ply as obtained from (9.20) or (9.31).
Three-Ply Plate

Let

\[ S = \text{maximum bending stress in a face ply} \]
\[ S_2 = \text{maximum bending stress in the center ply}. \]

\( E_x \) and \( E_y \) refer to the values of \( E \) in the face plies as before.

In accordance with (9.19)

\[ S = - \frac{E_x h}{2\lambda} \frac{d^2 W}{dx^2} \]
\[ S_2 = - \frac{E_y h}{6\lambda} \frac{d^2 W}{dx^2} \]

In the last equation \( E_y \) is the value of \( E \) in a direction parallel to the \( y \)-axis in a face ply and therefore the value of \( E \) in a direction parallel to the \( x \)-axis in the center ply. It follows that

\[ S_2 = \frac{S}{3} \frac{E_y}{E_x} \]  
(9.36)

Five-Ply Plate

Let

\[ S = \text{maximum bending stress in a face ply} \]
\[ S_2 = \text{maximum bending stress in an adjacent ply}. \]

\( E_x \) and \( E_y \) refer to the values of \( E \) in the face plies.

In accordance with (9.19)

\[ S = - \frac{E_x h}{2\lambda} \frac{d^2 W}{dx^2} \]
\[ S_2 = - \frac{3E_y h}{10\lambda} \frac{d^2 W}{dx^2} \]
Then

\[ S_2 = \frac{3}{5} \frac{E_y}{E_x} \, S \quad (9.37) \]

Tables 43 to 47, inclusive, contain a comparison of the results obtained by using the approximate formulas for load, maximum bending stress and direct stress with the results of calculations based on the exact formulas. In these tables \( Q \) denotes the uniform load in pounds per square foot, \( g \) a direct stress, and \( S \) a maximum bending stress. In calculating \( S \) the factor \( \alpha \) in equation (9.31) was taken to be 4.4. More accurate values of \( S \) would have been obtained by taking \( \alpha \) from the curve of figure 27.

The letters \( W_0 \) and \( h \) denote, respectively, the central deflection and the thickness of the plate.

The subscript \( \Omega \) denotes a result calculated by an approximate formula. If the subscript \( \Omega \) does not appear, the result was obtained by the exact method. The numerical subscripts 1 and 2 refer to the plies in which the stress is calculated, the subscript 1 referring to a face ply and 2 to a ply just under a face ply. The stresses are calculated in the ply for which the total stress, direct stress, plus maximum bending stress is greatest. This does not necessarily imply that failure occurs in this ply instead of in other plies. In the calculations for the plywood plates the elastic constants of spruce were used. For the steel plate \( E = 3 \times 10^7 \) pounds per square inch and \( \sigma = 0.3 \). The width of all plates is 48 inches. The thickness of the three-ply plates and of the steel plate is 3/8 inch, that of the five-ply plates is 5/8 inch.
Appendix 10.—Infinite Strip (Long Narrow Plate). Large Deflections. Uniformly Distributed Load. Edges Clamped

In this case the constants of equation (9.8) are to be determined to satisfy the conditions \( \gamma = 0, \frac{\partial \gamma}{\partial x} = 0 \) when \( x = 0 \) and when \( x = a \). It is again assumed that the edges of the plate are restrained from moving inward.

It is found that

\[
\gamma = \frac{px(a-x)}{2k^2D_t} - \frac{pa}{2k^3D_i} \left[ \cosh \frac{ka}{2} - \cosh k \left( x - \frac{a}{2} \right) \right] \sinh \frac{ka}{2}
\]  
(10.1)

The relation connecting \( p \) and \( q \) is found as in the case of a strip with simply supported edges by equating the two expressions in (9.10) and (9.11) for the integral of \( E_{xx} \) over the width of the strip.

Using (10.1) the following relation corresponding to (9.12) is obtained

\[
\frac{\lambda a g}{E_a} = \frac{p^2}{2k^7D_i^2} \left\{ 2ka + \frac{5}{3} \left( \frac{ka}{2} \right)^3 - \left( \frac{ka}{2} \right)^3 \coth^2 \frac{ka}{2} \right. \\
- \left. 3 \left( \frac{ka}{2} \right)^2 \coth \frac{ka}{2} \right\}
\]  
(10.2)

Using (9.7) and (2.14) this equation can be solved for the quantity \( P \) as defined by (9.13), in terms of the quantity \( \eta = ka/2 \).

It is found that

\[
P = \frac{4E_i}{3\lambda E_L} \left( \frac{E_i}{E_a} \right)^{\frac{1}{2}} K(\eta)
\]  
(10.3)

where

\[
K(\eta) = \frac{\eta^4}{(12 - 6\eta \coth \eta - 3\eta^2 \coth^2 \eta + 5 \eta^2)^{\frac{1}{2}}}
\]  
(10.4)

From (10.1) the deflection on the central line, \( x = a/2 \) is found to be given by

\[
\frac{\gamma_0}{h} = \frac{3\lambda P E_L}{8E_i} M(\eta)
\]  
(10.5)
where

\[ M(\eta) = \frac{1}{\eta^3} \left( \eta - 2 \tanh \frac{\eta}{2} \right) \]  \hspace{1cm} (10.6)

From (9.7) it is found as in the case of the plate with simply supported edges that the mean direct stress is expressed in terms of \( \eta \) by the equation

\[ g = \frac{E_I}{3\lambda} \frac{h^2}{a^2} \eta^2 \]  \hspace{1cm} (10.7)

The bending stress on the face \( z = -h/2 \) is calculated by the use of (9.19) and found to be

\[ \sigma_x = \frac{3PE_I E_x}{2E_I} \frac{h^2}{a^2} \left[ -\frac{1}{\eta^2} + \frac{1}{\eta} \frac{\cosh k(x-\frac{a}{2})}{\sinh \eta} \right] \]

This stress is a maximum at the edges \( x = 0 \) and \( x = a \). Denoting this maximum bending stress in a face ply by \( S \) we find that

\[ S = \frac{3PE_I E_x}{2E_I} \frac{h^2}{a^2} N(\eta) \]  \hspace{1cm} (10.8)

where

\[ N(\eta) = \left( \eta \coth \eta - 1 \right) / \eta^2 \]  \hspace{1cm} (10.9)

The calculations of the central deflection, mean direct stress and maximum bending stress in a face ply are to be carried out essentially as in the case of a plate with simply supported edges. The value of the parameter \( \eta \) associated with a given load is to be found from (10.3). Values of the functions \( K(\eta), M(\eta), \) and \( N(\eta) \) are given in table 42.

**Approximate Formulas**—As in the case of a plate with simply supported edges it is possible to replace the exact formulas just obtained for a uniformly loaded plate with clamped edges by very accurate approximate formulas connecting the load and the stresses with the deflection at the center.
From equations (10.3) and (10.5)

\[
\frac{w_0}{h} = \eta H(\eta) \left(\frac{E_i}{E_a}\right)^{\frac{1}{2}}
\]

where

\[
H(\eta) = \frac{K(\eta)M(\eta)}{2 \eta}
\]

The function \( H(\eta) \) is approximately constant. It can be replaced by the number 0.366 with an error of less than 2.5 percent for the range of values of \( \eta \) in which we are interested. Hence the following linear relation between \( \frac{w_0}{h} \) and \( \eta \) holds approximately:

\[
\frac{w_0}{h} = 0.366 \left(\frac{E_i}{E_a}\right)^{\frac{1}{2}} \eta
\]

(10.10)

It follows from (10.10) and (10.7) that

\[
g = \frac{2.488 E_a}{\lambda} \left(\frac{h}{a}\right)^2 \left(\frac{w_0}{h}\right)^2
\]

(10.11)

From (10.5) we obtain for small values of \( \eta \), using the Maclaurin’s series for \( \tanh \frac{\eta}{2} \):

\[
\frac{w_0}{h} = \frac{\lambda P E_L}{32 E_i} \left(1 - \frac{\eta^2}{10}\right)
\]

(10.12)

On comparing the approximate expression (10.12) with the exact expression (10.5) it is found that a better approximation is secured for the range of values in which we are interested if the factor \( 1 - \frac{\eta^2}{10} \) in (10.12) is replaced by \( 1 - \frac{\eta^2}{10.4} \). Solving the resulting equation for \( P \) we have approximately

\[
P = \frac{32 E_i}{\lambda E_L} \frac{w_0}{h} + \frac{4 E_i}{1.3 \lambda E_L} \eta^2 \frac{w_0}{h}
\]

The behavior of the function \( H(\eta) \) is shown by the following table of values:

<table>
<thead>
<tr>
<th>( \eta )</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H(\eta) )</td>
<td>0.3698</td>
<td>0.3696</td>
<td>0.3682</td>
<td>0.3662</td>
<td>0.3638</td>
<td>0.3622</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \eta )</th>
<th>12</th>
<th>14</th>
<th>16</th>
<th>18</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H(\eta) )</td>
<td>0.3610</td>
<td>0.3599</td>
<td>0.3590</td>
<td>0.3586</td>
<td>0.3580</td>
</tr>
</tbody>
</table>
On substituting for $\eta$ its expression in terms of $W_0/h$ from (10.10) we obtain

$$P = A \frac{W_0}{h} + B \left(\frac{W_0}{h}\right)^3$$

(10.13)

where

$$A = \frac{32 E_I}{\lambda E_L}$$

(10.14)

$$B = \frac{23 E_I}{\lambda E_L}$$

(10.15)

With the aid of (10.5) and (10.8) an approximate formula\(^36\) for the maximum bending stress in the face plies can be set up. From these equations we obtain

$$S = \alpha \frac{E_x}{\lambda} \left(\frac{h}{\alpha}\right)^2 \frac{W_0}{h}$$

(10.16)

where

$$\alpha = \frac{4 N(\eta)}{M(\eta)}$$

(10.17)

For small values of $\eta$ we find by expanding $N(\eta)$ and $M(\eta)$ in Maclaurin's series that

$$\alpha = 16 \left(1 + \frac{\eta^2}{30}\right)$$

(10.18)

On comparing the values of $\alpha$ given by (10.18) corresponding to the range of values of $\eta$ in which we are interested, with those given by the exact formula (10.17), we are led to replace the factor $1/30$ in (10.18) by $1/40$.

\(^{36}\)An alternative procedure is explained immediately after equation (10.22)
Then
\[ \alpha = 16(1 + 0.025 \eta^2) \]  \hspace{1cm} (10.19)

The expression for \( S \) becomes
\[ S = \frac{E_x}{\lambda} \left(16 + 0.4 \eta^2\right) \left(\frac{h}{a}\right)^2 \frac{w_0}{h} \]

On expressing \( \eta \) in terms of \( w_0/h \) by means of (10.10) we obtain
\[ S = C \frac{w_0}{h} + D \left(\frac{w_0}{h}\right)^3 \]  \hspace{1cm} (10.20)

where
\[ C = \frac{16 E_x}{\lambda} \left(\frac{h}{a}\right)^2 \]  \hspace{1cm} (10.21)
\[ D = \frac{2.98 E_x E_a}{\lambda E_1} \left(\frac{h}{a}\right)^2 \]  \hspace{1cm} (10.22)

Instead of using formula (10.20) in whose derivation considerable approximation is involved, the maximum bending stress in a face ply can be calculated directly from equation (10.16) with the aid of a curve giving \( \alpha \) as a function of \( \eta \). This curve which is the graph of (10.17) is given in figure 28. In using this curve the value of \( \eta \) associated with a given deflection is to be found from equation (10.10). The only approximation involved in the process is that contained in equation (10.10) in which the error is small.

After the maximum bending stress in a face ply has been calculated by one of the processes just described, the corresponding stress in other plies can be calculated with the aid of (9.36) or (9.37). Formulas (9.33) and (9.35) can be used to calculate the direct stress in a given ply from the mean direct stress \( \sigma \). In all of these formulas the plies are taken to be of equal thickness.

Tables 48 to 52, inclusive, contain a comparison of results calculated by the exact and approximate methods. The dimensions of the plates and the notation are given in appendix 9 in connection with the description of tables 43 to 47. The maximum bending stress was calculated by using (10.20). It would have been better to use (10.16), taking the value of \( \alpha \) from the curve of figure 28 as explained above.
Appendix 11.—Rectangular Plate (Large Deflections). Uniformly Distributed Load. Edges Simply Supported. Approximate Method

An approximate formula for the deflection $W_0$ at the center of the plate will be obtained by assuming the following expressions\(^{37}\) for the deflection $W$ and the components $u$ and $v$ parallel to the $X$- and $Y$-axes of the displacement of points in the middle surface of the plate. It is assumed that the edges of the plate are restrained from moving inward.

When $0 < y < c$ \hspace{1cm} (See figure 4),

\[
W = W_0 \sin \frac{\pi x}{a} \sin \frac{\pi y}{2c} \\
u = c_1 \sin \frac{2\pi x}{a} \sin \frac{\pi y}{2c} \\
v = c_2 \sin \frac{\pi x}{a} \sin \frac{\pi y}{c}
\]  \hspace{1cm} (11.1)

When $c < y < b - c$

\[
W = W_0 \sin \frac{\pi x}{a} \\
u = c_1 \sin \frac{2\pi x}{a} \sin \frac{\pi y}{a} \\
v = 0
\]  \hspace{1cm} (11.2)

Expressions corresponding to (11.1) are assumed for the region $b - c < y < b$ \hspace{1cm} Further let

\[
c = r a/2
\]  \hspace{1cm} (11.3)

\(^{37}\)Expressions corresponding to (11.1) were used by A. and L. Föppl—Drang and Zwang I p. 227. For plates which are not square it seems best to choose the forms (11.2) for the central portions.
The state of strain in the plate is made up of two parts, one associated with the bending stresses and given by (2.2), the other associated with the direct stresses and given by (8.1). The corresponding potential energies of deformation will be denoted by \( V_b \) and \( V_d \). Each of these is calculated as the sum of two parts arising from the end and middle portions of the plate, respectively. These parts will be denoted by the additional subscripts \( E \) and \( M \) respectively. Thus

\[
V_b = V_{be} + V_{bm} \\
V_d = V_{de} + V_{dm}
\]  

(11.4)

To these potential energies we have to add the change in the potential energy of the load, due to the deflection. This will be denoted by \( V_\ell \).

The parameters \( V_0, C_1, C_2, \) and \( T \) (or \( C \)) occurring in (11.1) and (11.2) are to be determined in such a way that the total potential energy, \( V = V_b + V_d + V_\ell \), of the system will be a minimum.

The expressions for \( V_\ell \) and \( V_b \) were calculated in appendix 3 (see (3.24), (3.25), and (3.26)).

The strain energy per unit area associated with the state of strain (8.1) is equal to

\[
\frac{h}{2} \left[ \overline{X}_x' e_{xx}' + \overline{Y}_y' e_{yy}' + \overline{X}_y' e_{xy}' \right] \\
= \frac{h}{2\lambda} \left[ E_a e_{xx}'^2 + 2 E_L \sigma_{xx} e_{yy}' + E_b e_{yy}'^2 + \lambda \mu_{LT} e_{xy}'^2 \right] 
\]  

(11.5)

where \( \overline{X}_x', \overline{Y}_y' \) and \( \overline{X}_y' \) are the mean direct stress components, the average being taken over the thickness of the plate. In the reduction to the second form the relations (8.3) and (8.4) have been used.

Using (8.1) the strain components are calculated for the assumed displacement and deflection as given by (11.1) and (11.2).
and substituted in the expression (11.5) for the strain energy per unit area, which is then integrated over the appropriate portions of the plate with the following results:

\[
V_{de} = \frac{hE_t}{2\lambda} \left[ A_1 \omega_0^4 + B_1 c_1 \omega_0^2 + C_1 c_2 \omega_0^2 + D_1 c_1^2 + F_1 c_2^2 + G_1 c_1 c_2 \right]
\]  \hspace{1cm} (11.6)

\[
V_{dm} = \frac{hE_l}{2\lambda} \left[ A_2 \omega_0^4 + B_2 c_1 \omega_0^2 + D_2 c_2^2 \right]
\]  \hspace{1cm} (11.7)

where

\[
A_1 = \frac{\pi^4}{256 a^2 \tau} \left( 4 \kappa_a \tau^2 + \frac{9 \kappa_b}{\tau^2} + 2 \sigma_{tl} + 4 \lambda \nu \right)
\]  \hspace{1cm} (11.8)

\[
B_1 = \frac{\pi^4}{3a} \left( 2 \kappa_a \tau + \frac{\lambda \nu - \sigma_{tl}}{\tau} \right)
\]  \hspace{1cm} (11.9)

\[
C_1 = \frac{\pi^4}{3a} \left( 2 \kappa_b \frac{\sigma_{tl}}{\tau^2} + \lambda \nu - \sigma_{tl} \right)
\]  \hspace{1cm} (11.10)

\[
D_1 = \pi^4 \left( \kappa_a \tau + \frac{\lambda \nu}{4\tau} \right)
\]  \hspace{1cm} (11.11)

\[
F_1 = \pi^4 \left( \frac{\kappa_b}{\tau} + \frac{\lambda \nu \tau}{4} \right)
\]  \hspace{1cm} (11.12)

\[
G_1 = \frac{32}{9} \left( \lambda \nu + \sigma_{tl} \right)
\]  \hspace{1cm} (11.13)
\[ A_2 = \frac{3\pi^4 \kappa_a}{32a^2} (k - \tau) \]  \hspace{2cm} (11.14)

\[ B_2 = \frac{\pi^3 \kappa_a}{2a} (k - \tau) \]  \hspace{2cm} (11.15)

\[ D_2 = 2\pi^2 \kappa_a (k - \tau) \]  \hspace{2cm} (11.16)

and

\[ \kappa_a = \frac{E_a}{E_L}, \quad \kappa_b = \frac{E_b}{E_L}, \quad \nu = \frac{\mu_L}{E_L}, \quad k = \frac{b}{a} \]  \hspace{2cm} (11.17)

and \( \kappa_1, \kappa_2, \delta \) and \( \lambda \) have the same meanings as in (3.29).

The parameters \( \mathcal{W}_0, C_1, C_2 \) and \( \tau \) are to be so determined that

\[ V = V_2 + V_{be} + V_{bm} + V_{de} + V_{dm} \]  \hspace{2cm} (11.18)

shall be a minimum.

The determination of \( \tau \) by this method proves to be too difficult. Instead, the value of \( \tau \) will be taken to be the same as that given by (3.34) for a plate with small deflections. This procedure is discussed in section 11. Using this value of \( \tau \), \( V \) becomes a function of \( \mathcal{W}_0 \), \( C_1 \) and \( C_2 \).

Since \( V \) is to be a minimum as a function of \( \mathcal{W}_0 \), \( C_1 \), and \( C_2 \) its first partial derivatives with respect to these quantities must vanish. From the equations \( \frac{\partial V}{\partial C_1} = 0 \) and \( \frac{\partial V}{\partial C_2} = 0 \), the quantities \( C_1 \) and \( C_2 \) can be expressed in terms of \( \mathcal{W}_0 \). When these values of \( C_1 \) and \( C_2 \) are entered in the equation, \( \frac{\partial V}{\partial \mathcal{W}_0} = 0 \), the following relation between \( \mathcal{W}_0 \) and \( P \) is obtained:

\[ P = \frac{H}{h} \mathcal{W}_0 + Q \left( \frac{\mathcal{W}_0}{h} \right)^3 \]  \hspace{2cm} (11.19)

-90-
where

\[ P = \frac{pa^4}{E_i h^4} \quad (11.20) \]

\[ H = \frac{\pi^6}{192\lambda} \frac{K^2}{T^3} + \frac{\delta}{T} + \kappa_i (2k - \tau) \quad (11.21) \]

\[ Q = \frac{\pi^2}{2\lambda} \frac{A - \frac{B^2 F - BCG + C^2 D}{4 DF - G^2}}{T + \frac{\pi}{2} (k - \tau)} \quad (11.22) \]

\[ A = (A_1 + A_2) a^2, \quad B = (B_1 + B_2) a, \]
\[ C = C_i a, \quad D = D_1 + D_2, \quad F = F_i, \quad G = G_i \quad (11.23) \]

When \( k = \infty \), that is, when the plate is an infinitely long strip with edges parallel to the Y-axis it is readily found that (11.21) and (11.22) become:

\[ H = \frac{\pi^6 \kappa_i}{48\lambda} \quad (11.24) \]

\[ Q = \frac{\pi^5 \kappa_i}{16\lambda} \quad (11.25) \]
Appendix 12.—Rectangular Plate (Large Deflections). Uniformly Distributed Load. Edges Clamped. Approximate Method

Using the assumed forms (12.1) and (12.2) of section 12 for the deflection and the displacement we obtain by the procedure of section 11 and appendix 11 the following approximate formula

\[ P = H \frac{w_0}{h} + Q \left( \frac{w_0}{h} \right)^3 \]  

(12.3)

where

\[ H = \frac{\pi^4}{12\lambda} \left( \frac{3\kappa_a^2}{\frac{\delta}{\lambda}} + \frac{\delta}{\lambda^2} + \kappa_a (8k - 5\tau) \right) \]  

(12.4)

\[ Q = \frac{8}{\lambda(2k-\tau)} \left[ A - \frac{B^2 F - BCG + C^2 D}{4DF - G^2} \right] \]  

(12.5)

where \( \tau \) is found from (4.30) appendix 4, \( k = b/a \) and

\[ A = \frac{\pi^4}{4096} \left( 105\kappa_a \tau + 105 \frac{\kappa_b}{\tau^3} + 25 \frac{\delta}{\tau} \right) \]  

\[ + \frac{3\pi^4}{32} \kappa_a (k - \tau) \]  

(12.6)

\[ B = -\frac{\pi^2}{15} \left( 16\kappa_a \tau - \frac{4\sigma_T}{\tau} + \frac{2\lambda v}{\tau} \right) \]  

\[ - \pi^3 \kappa_a (k - \tau) \]  

(12.7)

\[ C = -\frac{\pi^2}{15} \left( 16 \frac{\kappa_b}{\tau^2} - 4\sigma_T + 2\lambda v \right) \]  

(12.8)

\[ D = \pi^2 \left( 4\kappa_a \tau + \frac{\lambda v}{4\tau} \right) + 8\pi^2 \kappa_a (k - \tau) \]  

(12.9)

\[ F = \pi^2 \left( \frac{4\kappa_b}{\tau} + \frac{\lambda v \tau}{4} \right) \]  

(12.10)
\[ G = \frac{128}{225} (\sigma_{\text{TL}} + \lambda \nu) \quad (12.11) \]

The numbers \( \kappa_a, \kappa_b \) etc., are defined in appendix 11.

When \( k = b/a = \infty \) it is readily found that (12.4) and (12.5) become:

\[ H = \pi^4 \kappa / 3 \lambda \quad (12.12) \]

\[ Q = \pi^4 \kappa_a / 4 \lambda \quad (12.13) \]
Appendix 13.—General Notation

Choice of Axes and Designation of Type of Plate

In figure 1 with the choice of axes shown, the load is considered to be applied to the upper surface producing a deflection in the direction of the positive Z-axis. In figure 2 only the XY-plane is shown. The conventions as to signs of bending moments will be explained in appendix 2 in connection with figure 29. In using these conventions in connection with figure 2, the Z-axis is to be thought of as drawn outward from the paper and the load as applied on the underside. The abbreviations 3X, 5X, ..., denote three-ply, five-ply, ... plates with the grain of face plies parallel to the X-axis. The abbreviations 3Y, 5Y, ..., denote three-ply, five-ply, ... plates with the grain of face plies parallel to the Y-axis.

Symbols

\[ a, b \quad -- \text{lengths of sides of rectangular plate as shown in figure 2.} \quad a \leq b \]

\[ D_1, D_2 \quad -- \text{Coefficients of flexural rigidity defined by the equations} \]

\[ D_1 = \frac{E_1 h^3}{12 \lambda}, \quad D_2 = \frac{E_2 h^3}{12 \lambda} \]

\[ e_{xx}, e_{xy} \quad -- \text{Components of strain. (Love, A. E. H., The Mathematical Theory of Elasticity, Art. 8.)} \]

\[ E_x, E_y, E_z \quad -- \text{Young's moduli at a given point associated with tension or compression parallel to the X, Y, or Z-axes, respectively.} \]

\[ E_1 \quad -- \text{Mean Young's modulus in bending under a couple whose axis is perpendicular to the XZ-plane. See equation (2.18) of appendix 2 and page 7.} \]

\[ E_2 \quad -- \text{Mean Young's modulus in bending under a couple whose axis is perpendicular to the YZ-plane. See equation (2.19) of appendix 2 and page 8.} \]
\( E_a \) -- Mean Young's modulus in stretching in a direction parallel to the X-axis. See equation (8.4) of appendix 8.

\( E_b \) -- Mean Young's modulus in stretching in a direction parallel to the Y-axis. See equation (8.4) of appendix 8.

\( g \) -- Mean direct stress \( X' \) in sections 9 and 10.

\( h \) -- Thickness of plate.

\( k = \frac{b}{a} \) -- Ratio of length to breadth of rectangular plate.

\( \sqrt[k]{\frac{gh}{D_i}} \) -- In sections 9 and 10.

\( K = \left( \frac{E_l \sigma_{\ell}}{\lambda} + 2 \mu_{LL} \right) \frac{h^3}{12 \lambda} \)

\( K(\eta) \) -- a functional symbol in section 10.

\( L, R, T \) -- Subscripts denoting directions parallel to the longitudinal, radial, and tangential directions, respectively, in wood; that is, \( L \) denotes the direction parallel to the grain, \( R \) the direction at right angles to the annual rings considered to be plane, and \( T \) the direction parallel to the annual rings and perpendicular to the grain of the wood. See figure 3.

\( m_x \) -- Bending moment per unit length of a vertical section of the plate perpendicular to the X-axis.

\( m_y \) -- Bending moment per unit length of a vertical section of the plate perpendicular to the Y-axis.

\( m_{xy} \) -- Twisting moment per unit length of a vertical section of the plate perpendicular to either the X- or the Y-axis.

\( p \) -- Load per unit area.

\[ p = \frac{p a^4}{E_l h^4} \] -- In case of uniformly distributed load.

(In sections 3, 4, 9, 10, 11, and 12 and in corresponding appendices.)
--- Load in case of a load concentrated at a point or total load in case of load distributed over a small area. (In sections 5, 6, 7 and in the corresponding appendixes.)

--- Subscript denoting radial direction in wood.

--- Maximum bending stress in face plies.

--- Subscript denoting tangential direction in wood.

--- Displacement parallel to the X-axis of a point in the plate.

--- Displacement parallel to the Y-axis of a point in the plate.

--- Deflection of a point in the middle surface of the plate; that is, the displacement of this point parallel to the Z-axis.

--- Deflection at the center of a plate.

--- Deflection on the center line of an infinitely long plate.


\[ \delta = \varepsilon b \]

\[ \gamma_n = \lambda_n \rho \]

\[ \delta_n = \lambda_n \sigma \]

\[ \delta = 2\sigma_{\tau I} + 4\lambda \nu \]

\[ \varepsilon = \left( \frac{D_1}{D_2} \right)^{1/4} = \left( \frac{E_1}{E_2} \right)^{1/4} \]

\[ \eta = \varepsilon y \]

\[ \eta = \frac{k a}{2} \text{ in sections 9 and 10.} \]
\[ \kappa = \frac{K}{(D_1 D_2)^{1/2}} \]

\( \kappa (\eta) \) -- A functional symbol in section 9.

\[ \kappa_1 = \frac{E_1}{E_L} \]

\[ \kappa_2 = \frac{E_2}{E_L} \]

\[ \kappa_a = \frac{E_a}{E_L} \]

\[ \kappa_b = \frac{E_b}{E_L} \]

\[ \lambda = 1 - \sigma_{LT} \sigma_{TL} \quad \text{for flat-grain plies} \]

\[ \lambda_n = \frac{n \pi}{a}, \quad n \text{ an integer} \]

\( \mu_{xy} \) -- Modulus of rigidity associated with the shearing strain \( \epsilon_{xy} \) in a plane parallel to the XY-plane. There are two other moduli of rigidity, \( \mu_{yz} \) and \( \mu_{zx} \).

\[ \nu = \frac{\mu_{LT}}{E_L} \]

\[ \rho = \left( \frac{1 + \kappa^2}{2} \right)^{1/2} \]

\[ \sigma = \left( \frac{1 - \kappa^2}{2} \right)^{1/2} \]

\( \sigma \) = Poisson's ratio in isotropic material. Not related to \( \sigma \) defined in preceding line.
\[ \sigma_{xy} \quad \text{--- Poisson's ratio associated with a tensile stress parallel to the X-axis and contraction parallel to the Y-axis. There are five other Poisson's ratios similarly defined:} \]
\[
\sigma_{yx}, \sigma_{xz}, \sigma_{zx}, \sigma_{yz}, \text{ and } \sigma_{zy}
\]

\[ \sigma_1 = \frac{E_L}{E_1} \sigma_{TL} \quad \text{for flat-grain plies.} \]

\[ \sigma_2 = \frac{E_L}{E_2} \sigma_{TL} \quad \text{for flat-grain plies.} \]

\[ \tau \quad \text{In all sections except section 6 this symbol is defined by (3.18). In section 6 it is defined by (6.7).} \]
Table 1.—Elastic constants of timber

<table>
<thead>
<tr>
<th></th>
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<tbody>
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</tr>
<tr>
<td></td>
<td>L. R. T.</td>
<td>L. R. T.</td>
<td>LT. LR. RT.</td>
<td>$\sigma_{LR} \sigma_{LT} \sigma_{RT}$</td>
<td>$\mu_1 \mu_2 \mu_3$</td>
</tr>
<tr>
<td>Spruce.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Density 27 lbs. per c.f.t.</td>
<td>$E \times 10^6$</td>
<td>1.95 0.13 0.07</td>
<td>— — —</td>
<td>— — —</td>
<td>— — —</td>
</tr>
<tr>
<td>Moisture 12.2 %</td>
<td>5900 Lbs./sq.in.</td>
<td>7000 Lbs./sq.in.</td>
<td>7000 900</td>
<td>18000 800 600</td>
<td>1100 1200 300</td>
</tr>
<tr>
<td>Light Mahogany.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Density 33 lbs. per c.f.t.</td>
<td>$E \times 10^6$</td>
<td>1.8 0.14 0.07</td>
<td>— — —</td>
<td>— — —</td>
<td>— — —</td>
</tr>
<tr>
<td>Moisture 13.4 %</td>
<td>5640 Lbs./sq.in.</td>
<td>6500 Lbs./sq.in.</td>
<td>1920 1000</td>
<td>17800 1080 650</td>
<td>1050 1540 500</td>
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<tr>
<td>Ash.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Density 50 lbs. per c.f.t.</td>
<td>$E \times 10^6$</td>
<td>2.18 0.238 0.14</td>
<td>— — —</td>
<td>— — —</td>
<td>— — —</td>
</tr>
<tr>
<td>Moisture 13.6 %</td>
<td>3800 Lbs./sq.in.</td>
<td>7300 Lbs./sq.in.</td>
<td>400 170</td>
<td>3800 2270 1080</td>
<td>2000 2000 600</td>
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<tr>
<td>Walnut.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Density 37 lbs. per c.f.t.</td>
<td>$E \times 10^6$</td>
<td>1.63 0.172 0.092</td>
<td>— — —</td>
<td>— — —</td>
<td>— — —</td>
</tr>
<tr>
<td>Moisture 11.0 %</td>
<td>4200 Lbs./sq.in.</td>
<td>8480 Lbs./sq.in.</td>
<td>650 520</td>
<td>24000 1690 1080</td>
<td>2000 2000 500</td>
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Table 2.—Apparent moduli in bending of spruce plywood. Plies of equal thickness

<table>
<thead>
<tr>
<th>Number of plies</th>
<th>$E_1/E_L$</th>
<th>$E_2/E_L$</th>
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</thead>
<tbody>
<tr>
<td>3</td>
<td>0.964</td>
<td>0.072</td>
</tr>
<tr>
<td>5</td>
<td>0.799</td>
<td>0.236</td>
</tr>
<tr>
<td>7</td>
<td>0.722</td>
<td>0.314</td>
</tr>
<tr>
<td>9</td>
<td>0.677</td>
<td>0.359</td>
</tr>
</tbody>
</table>

Table 3.—Apparent moduli in bending of Douglas-fir plywood. Plies of equal thickness

<table>
<thead>
<tr>
<th>Number of plies</th>
<th>$E_1/E_L$</th>
<th>$E_2/E_L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.965</td>
<td>0.092</td>
</tr>
<tr>
<td>5</td>
<td>0.804</td>
<td>0.253</td>
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<tr>
<td>7</td>
<td>0.728</td>
<td>0.330</td>
</tr>
<tr>
<td>9</td>
<td>0.664</td>
<td>0.373</td>
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</table>
Table 4.—Plates of type 3Y. Uniform load. Edges simply supported

<table>
<thead>
<tr>
<th>( k )</th>
<th>( \tau )</th>
<th>( \alpha ) (approximate)</th>
<th>( \alpha ) (exact)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>1.5</td>
<td>0.575</td>
<td>0.6213</td>
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<tr>
<td>1.5</td>
<td>2.0</td>
<td>1.110</td>
<td>1.0900</td>
</tr>
<tr>
<td>2.0</td>
<td>3.0</td>
<td>1.895</td>
<td>1.8360</td>
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<tr>
<td>4.0</td>
<td>3.5</td>
<td>2.125</td>
<td>2.1320</td>
</tr>
<tr>
<td>( \infty )</td>
<td>2.157</td>
<td>2.1480</td>
<td></td>
</tr>
</tbody>
</table>

Table 5.—Plates of type 3X. Uniform load. Edges simply supported

<table>
<thead>
<tr>
<th>( k )</th>
<th>( \tau )</th>
<th>( \alpha ) (approximate)</th>
<th>( \alpha ) (exact)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.95</td>
<td>0.1581</td>
<td>0.1541</td>
</tr>
<tr>
<td>1.5</td>
<td>0.96</td>
<td>0.1598</td>
<td>0.1654</td>
</tr>
<tr>
<td>2.0</td>
<td>0.97</td>
<td>0.1604</td>
<td>0.1629</td>
</tr>
<tr>
<td>3.0</td>
<td>0.97</td>
<td>0.1606</td>
<td>0.1604</td>
</tr>
<tr>
<td>4.0</td>
<td>0.97</td>
<td>0.1607</td>
<td>0.1604</td>
</tr>
<tr>
<td>( \infty )</td>
<td>0.1611</td>
<td>0.1604</td>
<td></td>
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</tbody>
</table>
Table 6.—Plates of type 5Y. Uniform load. Edges simply supported

<table>
<thead>
<tr>
<th>k</th>
<th>τ</th>
<th>α (approximate)</th>
<th>α (exact)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>(See table 7)</td>
<td></td>
<td></td>
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</table>

Table 7.—Plates of type 5X. Uniform load. Edges simply supported

<table>
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<th>α (exact)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
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<td>0.1585</td>
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<td>1.34</td>
<td>0.2012</td>
<td>0.2011</td>
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<td>0.1983</td>
<td>0.2048</td>
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<td>1.31</td>
<td>0.1960</td>
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<td>0.1934</td>
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</table>

Table 8.—Plates of type 7X. Uniform load. Edges simply supported

<table>
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<th>α (approximate)</th>
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<td>0.2207</td>
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<td>1.43</td>
<td>0.2180</td>
</tr>
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<td>4.0</td>
<td>1.43</td>
<td>0.2172</td>
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<td>0.2152</td>
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Table 9.--- Isotropic plates. (σ = 0.25) Uniform load. Edges simply supported

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<th>α (exact)</th>
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<td>0.1139</td>
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Table 10.---The factor γ of formula (3.45)

<table>
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<th>γ</th>
<th>β/a</th>
<th>γ</th>
<th>β/a</th>
<th>γ</th>
<th>β/a</th>
<th>γ</th>
<th>β/a</th>
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</thead>
<tbody>
<tr>
<td>0.50</td>
<td>0.06</td>
<td>0.750</td>
<td>0.490</td>
<td>1.000</td>
<td>0.765</td>
<td>1.250</td>
<td>0.897</td>
<td>1.500</td>
</tr>
<tr>
<td>2.25</td>
<td>1.03</td>
<td>2.500</td>
<td>1.038</td>
<td>2.750</td>
<td>1.040</td>
<td>3.000</td>
<td>1.036</td>
<td>3.500</td>
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Table 11.—Results of tests of uniformly loaded plates.
Simply supported edges

<table>
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<tr>
<th>Panel and type</th>
<th>b</th>
<th>a</th>
<th>h</th>
<th>E₁ 10⁶</th>
<th>E₂ 10⁶</th>
<th>Side A: p/w₀</th>
<th>Side B: p/w₀</th>
<th>Average: p/w₀</th>
<th>β: α⁻¹</th>
<th>γ₀obs</th>
<th>γthet</th>
<th>(from curve)</th>
<th>γ₀theor</th>
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</thead>
<tbody>
<tr>
<td>4-3X</td>
<td>90.5</td>
<td>42.5</td>
<td>0.372</td>
<td>1.582</td>
<td>0.190</td>
<td>0.2190</td>
<td>0.2219</td>
<td>0.220</td>
<td>3.66</td>
<td>0.734</td>
<td>1.022</td>
<td>0.718</td>
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</tr>
<tr>
<td>5-5X</td>
<td>138.5</td>
<td>42.5</td>
<td>0.499</td>
<td>1.398</td>
<td>0.462</td>
<td>0.3501</td>
<td>0.3399</td>
<td>0.345</td>
<td>4.30</td>
<td>0.999</td>
<td>1.005</td>
<td>0.994</td>
<td></td>
</tr>
<tr>
<td>5-5X</td>
<td>90.5</td>
<td>42.5</td>
<td>0.500</td>
<td>1.406</td>
<td>0.443</td>
<td>0.4261</td>
<td>0.3837</td>
<td>0.405</td>
<td>2.84</td>
<td>0.851</td>
<td>1.039</td>
<td>0.829</td>
<td></td>
</tr>
<tr>
<td>5-5X</td>
<td>42.5</td>
<td>42.5</td>
<td>0.498</td>
<td>1.282</td>
<td>0.472</td>
<td>0.5480</td>
<td>0.5120</td>
<td>0.530</td>
<td>1.28</td>
<td>0.592</td>
<td>0.735</td>
<td>0.754</td>
<td></td>
</tr>
<tr>
<td>7-5Y</td>
<td>138.5</td>
<td>42.5</td>
<td>0.609</td>
<td>1.436</td>
<td>0.607</td>
<td>0.607</td>
<td>0.3290</td>
<td>0.348</td>
<td>1.39</td>
<td>0.895</td>
<td>1.039</td>
<td>0.862</td>
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<tr>
<td>5-5Y</td>
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<td>42.5</td>
<td>0.607</td>
<td>1.476</td>
<td>0.616</td>
<td>0.7790</td>
<td>0.7830</td>
<td>0.784</td>
<td>2.99</td>
<td>0.930</td>
<td>1.037</td>
<td>0.929</td>
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<tr>
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<td>42.5</td>
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<td>1.561</td>
<td>0.610</td>
<td>0.610</td>
<td>0.9720</td>
<td>0.986</td>
<td>0.979</td>
<td>0.32</td>
<td>0.288</td>
<td>0.310</td>
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</tr>
<tr>
<td>9-5Y</td>
<td>138.5</td>
<td>42.5</td>
<td>0.743</td>
<td>1.751</td>
<td>1.100</td>
<td>0.830</td>
<td>0.7440</td>
<td>0.714</td>
<td>2.97</td>
<td>0.355</td>
<td>1.037</td>
<td>0.824</td>
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<td>42.5</td>
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<td>0.753</td>
<td>1.1500</td>
<td>1.0220</td>
<td>1.086</td>
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<td>1.020</td>
<td>0.904</td>
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<tr>
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<td>42.5</td>
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<td>0.825</td>
<td>0.900</td>
<td>2.2310</td>
<td>2.0380</td>
<td>2.134</td>
<td>3.19</td>
<td>0.754</td>
<td>1.033</td>
<td>0.730</td>
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<tr>
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<td>42.5</td>
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<td>0.998</td>
<td>0.899</td>
<td>2.1700</td>
<td>2.2580</td>
<td>2.202</td>
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<td>1.030</td>
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Mean: ..................................... 0.803

1312
Table 12.—Plates of type 3Y. Uniform load. Edges clamped

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<th>α  (approximate)</th>
<th>α  (exact)</th>
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</thead>
<tbody>
<tr>
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<td>0.0943</td>
<td>0.943</td>
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<tr>
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<tr>
<td>2.00</td>
<td>2.46</td>
<td>0.3880</td>
<td>0.3795</td>
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<td>3.00</td>
<td>2.53</td>
<td>0.4055</td>
<td>0.404</td>
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<td>2.55</td>
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Table 13.—Plates of type 3X. Uniform load. Edges clamped

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<th>α  (exact)</th>
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<td>0.03058</td>
<td>0.03332</td>
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<td>0.70</td>
<td>0.03086</td>
<td>0.03279</td>
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<td>0.03295</td>
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<td>0.71</td>
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Table 14.—Plates of type 5Y. Uniform load. Edges clamped

<table>
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<th>α  (approximate)</th>
<th>α  (exact)</th>
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<tbody>
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<td>0.00250</td>
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Table 15.—Plates of type 5X. Uniform load. Edges clamped

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<th>$\alpha$ (exact)</th>
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<td>0.03890</td>
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<td>0.97</td>
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<td>0.04093</td>
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<td>0.98</td>
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<td>0.04119</td>
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Table 16.—Plates of type 7Y. Uniform load. Edges clamped

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<th>$\alpha$ (approximate)</th>
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<tbody>
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</tr>
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<td>1.25</td>
<td>1.25</td>
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</tr>
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<td>1.50</td>
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Table 17.—Plates of type 7X. Uniform load. Edges clamped

<table>
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<th>$\tau$</th>
<th>$\alpha$ (approximate)</th>
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</thead>
<tbody>
<tr>
<td>1.00</td>
<td>1.00</td>
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<td>1.05</td>
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Table 18.—Plates of type 9Y. Uniform load. Edges clamped

<table>
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<th>( \alpha ) (approximate)</th>
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</thead>
<tbody>
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<td>1.00</td>
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<td>1.25</td>
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<td>( \infty )</td>
<td>( \ldots \ldots )</td>
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Table 19.—Plates of type 9X. Uniform load. Edges clamped

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<th>( \alpha ) (approximate)</th>
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</thead>
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<td>1.00</td>
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<td>( \infty )</td>
<td>( \ldots \ldots )</td>
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</tbody>
</table>
Table 20.—Isotropic plates ( $\sigma = 0.25$). Uniform load. Edges clamped

<table>
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<th>$\alpha$ (approximate)</th>
<th>$\alpha$ (exact)</th>
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</thead>
<tbody>
<tr>
<td>1.00</td>
<td>1.00</td>
<td>0.01444</td>
<td>0.01406</td>
</tr>
<tr>
<td>1.25</td>
<td>1.14</td>
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<td>0.02032</td>
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<td>1.23</td>
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</table>

Table 21.—The factor $\gamma$ of formula \(4.35\)

<table>
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<th>$\beta/a$</th>
<th>$\gamma$</th>
<th>$\beta/a$</th>
<th>$\gamma$</th>
<th>$\beta/a$</th>
<th>$\gamma$</th>
<th>$\beta/a$</th>
<th>$\gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.500</td>
<td>0.065</td>
<td>0.750</td>
<td>0.285</td>
<td>1.000</td>
<td>0.575</td>
<td>1.250</td>
<td>0.835</td>
</tr>
<tr>
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<td>0.960</td>
<td>1.750</td>
<td>1.015</td>
<td>2.000</td>
<td>1.042</td>
<td>2.250</td>
<td>1.051</td>
</tr>
<tr>
<td>2.500</td>
<td>1.044</td>
<td>2.750</td>
<td>1.026</td>
<td>3.000</td>
<td>1.015</td>
<td>3.500</td>
<td>1.002</td>
</tr>
<tr>
<td>Panel and type</td>
<td>Panel</td>
<td>$b$</td>
<td>$a$</td>
<td>$h$</td>
<td>$E_1 \times 10^6$</td>
<td>$E_2 \times 10^6$</td>
<td>$\beta \gamma_{obs}$</td>
</tr>
<tr>
<td>---------------</td>
<td>-------</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----------------</td>
<td>-----------------</td>
<td>-------------------</td>
</tr>
<tr>
<td>4-3X</td>
<td>90.5</td>
<td>42.5</td>
<td>0.372</td>
<td>1.582</td>
<td>0.190</td>
<td>0.660</td>
<td>0.791</td>
</tr>
<tr>
<td>3X</td>
<td>42.5</td>
<td>42.5</td>
<td>0.380</td>
<td>1.475</td>
<td>0.138</td>
<td>0.362</td>
<td>0.510</td>
</tr>
<tr>
<td>5-5Y</td>
<td>90.5</td>
<td>42.5</td>
<td>0.499</td>
<td>1.188</td>
<td>0.188</td>
<td>0.475</td>
<td>0.513</td>
</tr>
<tr>
<td>5Y</td>
<td>42.5</td>
<td>42.5</td>
<td>0.498</td>
<td>1.204</td>
<td>0.204</td>
<td>0.938</td>
<td>0.957</td>
</tr>
<tr>
<td>6-5X</td>
<td>138.5</td>
<td>42.5</td>
<td>0.499</td>
<td>1.398</td>
<td>0.462</td>
<td>1.491</td>
<td>1.388</td>
</tr>
<tr>
<td>5X</td>
<td>90.5</td>
<td>42.5</td>
<td>0.500</td>
<td>1.406</td>
<td>0.443</td>
<td>1.293</td>
<td>1.349</td>
</tr>
<tr>
<td>5Y</td>
<td>42.5</td>
<td>42.5</td>
<td>0.498</td>
<td>1.282</td>
<td>0.472</td>
<td>1.021</td>
<td>1.190</td>
</tr>
<tr>
<td>7-5X</td>
<td>138.5</td>
<td>42.5</td>
<td>0.609</td>
<td>1.136</td>
<td>0.910</td>
<td>1.934</td>
<td>0.922</td>
</tr>
<tr>
<td>5Y</td>
<td>90.5</td>
<td>42.5</td>
<td>0.607</td>
<td>1.176</td>
<td>0.534</td>
<td>1.225</td>
<td>1.175</td>
</tr>
<tr>
<td>5Y</td>
<td>42.5</td>
<td>42.5</td>
<td>0.606</td>
<td>1.423</td>
<td>0.224</td>
<td>2.217</td>
<td>2.231</td>
</tr>
<tr>
<td>8-5X</td>
<td>138.5</td>
<td>42.5</td>
<td>0.610</td>
<td>1.561</td>
<td>0.492</td>
<td>2.650</td>
<td>2.430</td>
</tr>
<tr>
<td>5X</td>
<td>90.5</td>
<td>42.5</td>
<td>0.610</td>
<td>1.622</td>
<td>0.416</td>
<td>2.437</td>
<td>2.531</td>
</tr>
<tr>
<td>9-5Y</td>
<td>138.5</td>
<td>42.5</td>
<td>0.743</td>
<td>1.100</td>
<td>2.186</td>
<td>2.123</td>
<td>2.154</td>
</tr>
<tr>
<td>10-5X</td>
<td>138.5</td>
<td>42.5</td>
<td>0.748</td>
<td>1.208</td>
<td>0.753</td>
<td>3.310</td>
<td>3.078</td>
</tr>
<tr>
<td>11-7Y</td>
<td>138.5</td>
<td>42.5</td>
<td>0.995</td>
<td>0.825</td>
<td>0.900</td>
<td>5.580</td>
<td>5.510</td>
</tr>
<tr>
<td>12-7X</td>
<td>138.5</td>
<td>42.5</td>
<td>1.001</td>
<td>0.998</td>
<td>0.899</td>
<td>5.880</td>
<td>5.580</td>
</tr>
</tbody>
</table>

Mean: 1.403
Table 23.—Infinite strip of plywood of type 3X. Deflection and bending moments at points along the lines $y = 0$ and $x = 0.5a$ due to a concentrated load at the point $(0.5a, 0)$.

<table>
<thead>
<tr>
<th>$x/a$</th>
<th>0.0000</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w/(Pa^2/D_1)$</td>
<td>.0000</td>
<td>.0107</td>
<td>.0208</td>
<td>.0297</td>
<td>.0364</td>
<td>.0395</td>
<td>...</td>
</tr>
<tr>
<td>$m_x/P$</td>
<td>.0000</td>
<td>.0635</td>
<td>.1340</td>
<td>.2234</td>
<td>.3661</td>
<td>$\infty$</td>
<td>...</td>
</tr>
<tr>
<td>$m_y/P$</td>
<td>.0000</td>
<td>.0174</td>
<td>.0366</td>
<td>.0611</td>
<td>.1001</td>
<td>$\infty$</td>
<td>...</td>
</tr>
</tbody>
</table>

$x = 0.5a$

<table>
<thead>
<tr>
<th>$y/a$</th>
<th>.0000</th>
<th>.1</th>
<th>.2</th>
<th>.3</th>
<th>.4</th>
<th>.5</th>
<th>.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w/(Pa^2/D_1)$</td>
<td>.0395</td>
<td>.0333</td>
<td>.0237</td>
<td>.0149</td>
<td>.0084</td>
<td>.0041</td>
<td>.0016</td>
</tr>
<tr>
<td>$m_x/P$</td>
<td>$\infty$</td>
<td>.3846</td>
<td>.2351</td>
<td>.1418</td>
<td>.0779</td>
<td>.0368</td>
<td>.0131</td>
</tr>
<tr>
<td>$m_y/P$</td>
<td>$\infty$</td>
<td>.0267</td>
<td>-.0036</td>
<td>-.0141</td>
<td>-.0154</td>
<td>-.0126</td>
<td>-.0088</td>
</tr>
</tbody>
</table>
Table 24.—Infinite strip of plywood of type JX. Load uniformly distributed
over a small square whose center is at the point \( (0.5a, 0) \).
Deflection and bending moments along the lines \( y = 0 \) and
\( x = 0.5a \). Total load \( = P \). Sides of square are 0.1a

| \( x/a \) | 0.0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |... \\
| \( w/(Pa^2/D_1) \) | .0000 | .0104 | .0205 | .0295 | .0358 | .0381 |... \\
| \( m_x/P \) | .0000 | .0647 | .1390 | .2280 | .3990 | .5980 |... \\
| \( m_y/P \) | .0000 | .0170 | .0358 | .0583 | .0883 | .1114 |...

\[ \begin{align*}
\text{y} &= 0 \\
\text{x} &= 0.5a
\end{align*} \]

| \( y/a \) | 0.0000 | .1 | .2 | .3 | .4 | .5 | .6 \\
| \( w/(Pa^2/D_1) \) | .0381 | .0330 | .0236 | .0150 | .0085 | .0042 | .0016 \\
| \( m_x/P \) | .5980 | .3860 | .2370 | .1420 | .0785 | .0374 | .0136 \\
| \( m_y/P \) | .1114 | .0311 | -.0026 | -.0136 | -.0152 | -.0126 | -.0088 |

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Table 25.—Square plate of plywood of type 3X. Deflection and bending moments at points along the lines \( y = 0 \) and \( x = 0.5a \) due to a concentrated load at the center \((0.5a, 0)\). Edges simply supported.

<table>
<thead>
<tr>
<th>( x/a )</th>
<th>0.0000</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w/(Pa^2/D_{1}) )</td>
<td>0.0000</td>
<td>0.0109</td>
<td>0.0213</td>
<td>0.0303</td>
<td>0.0371</td>
<td>0.0402</td>
</tr>
<tr>
<td>( m_x/P )</td>
<td>0.0000</td>
<td>0.0646</td>
<td>0.1362</td>
<td>0.2264</td>
<td>0.3697</td>
<td>( \infty )</td>
</tr>
<tr>
<td>( m_y/P )</td>
<td>0.0000</td>
<td>0.0175</td>
<td>0.0369</td>
<td>0.0615</td>
<td>0.1006</td>
<td>( \infty )</td>
</tr>
</tbody>
</table>

\( x = 0.5a \)

<table>
<thead>
<tr>
<th>( y/a )</th>
<th>0.0000</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w/(Pa^2/D_{1}) )</td>
<td>0.0402</td>
<td>0.0340</td>
<td>0.0241</td>
<td>0.0148</td>
<td>0.0069</td>
<td>0.0000</td>
</tr>
<tr>
<td>( m_x/P )</td>
<td>( \infty )</td>
<td>0.3890</td>
<td>0.2385</td>
<td>0.1405</td>
<td>0.0648</td>
<td>0.0000</td>
</tr>
<tr>
<td>( m_y/P )</td>
<td>( \infty )</td>
<td>0.0281</td>
<td>-0.0007</td>
<td>-0.0086</td>
<td>-0.0066</td>
<td>0.0000</td>
</tr>
</tbody>
</table>
Table 26.—Square plate of plywood of type 3X. Deflection and bending moments at points along the lines \( y = 0 \) and \( x = 0.5a \). Load uniformly distributed over a small square area at the center \((0.5a, 0)\) of the plate. Edges simply supported. Sides of square = 0.1a.

<table>
<thead>
<tr>
<th>( y = 0 )</th>
<th>( x/a )</th>
<th>( 0.0 )</th>
<th>( 0.1 )</th>
<th>( 0.2 )</th>
<th>( 0.3 )</th>
<th>( 0.4 )</th>
<th>( 0.5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w/(Pa^2/D_1) )</td>
<td>.0000</td>
<td>.0105</td>
<td>.0205</td>
<td>.0295</td>
<td>.0359</td>
<td>.0383</td>
<td></td>
</tr>
<tr>
<td>( m_x/P )</td>
<td>.0000</td>
<td>.0650</td>
<td>.1390</td>
<td>.2290</td>
<td>.4000</td>
<td>.5990</td>
<td></td>
</tr>
<tr>
<td>( m_y/P )</td>
<td>.0000</td>
<td>.0170</td>
<td>.0360</td>
<td>.0570</td>
<td>.0890</td>
<td>.1100</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x = 0.5a )</th>
<th>( y/a )</th>
<th>( 0.000 )</th>
<th>( 1 )</th>
<th>( 2 )</th>
<th>( 3 )</th>
<th>( 4 )</th>
<th>( 5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w/(Pa^2/D_1) )</td>
<td>.383</td>
<td>.0332</td>
<td>.0238</td>
<td>.0147</td>
<td>.0070</td>
<td>.0000</td>
<td></td>
</tr>
<tr>
<td>( m_x/P )</td>
<td>.5990</td>
<td>.3880</td>
<td>.2350</td>
<td>.1400</td>
<td>.0650</td>
<td>.0000</td>
<td></td>
</tr>
<tr>
<td>( m_y/P )</td>
<td>.1100</td>
<td>.0320</td>
<td>-.0010</td>
<td>-.0090</td>
<td>-.0070</td>
<td>.0000</td>
<td></td>
</tr>
</tbody>
</table>
Table 27.—Factor $\gamma$ of (7.5). Concentrated load at center of rectangular plate of plywood. Edges simply supported

<table>
<thead>
<tr>
<th>Type</th>
<th>0.25</th>
<th>0.50</th>
<th>0.75</th>
<th>1.00</th>
<th>1.25</th>
<th>1.50</th>
<th>2.00</th>
<th>2.50</th>
<th>3.00</th>
<th>4.00</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3X : 0.0618 : 0.2536 : 0.5519 : 0.8129 : 0.9582 : 1.0101 : 1.0167 : 1.0049 : 1.0005 : 1.0000


¹Face plies one-half as thick as remaining plies.
Table 28.—Results of tests of concentrated loads on plywood plates. Simply supported edges

| Panel Type | \( b \) | \( a \) | \( h \) | \( E' \times 10^{-6} \) | \( E'' \times 10^{-6} \) | \( \gamma \) | \( E' \) | \( \frac{P}{b} \times 10^{-6} \) | \( \frac{B}{a} \) | \( \gamma_{\text{obs.}} \) | \( \gamma_{\text{theor.}} \) | \( \gamma_{\text{theor.}} \) |
|------------|------|------|------|----------------|----------------|------|------|----------------|---------|--------|--------|---------|---------|
| 4-3X       | 90.5 | 42.5 | 1.372| 1.582         | 0.190          | 0.862| 1.690| 7.760         | 3.60    | 0.886  | 1.000  | 0.886   | .970    |
| 3X         | 42.5 | 42.5 | 1.380| 1.175         | 0.183          | 0.862| 1.676| 8.300         | 1.68    | 0.999  | 1.030  | .970    |
| 5-5Y       | 138.5| 42.5 | 1.499| 1.444         | 1.190          | 1.813| 7.82 | 5.000         | 2.55    | 0.789  | 1.003  | .786    |
| 5Y         | 90.5 | 42.5 | 1.499| 1.438         | 1.187          | 1.813| 7.80 | 4.610         | 1.66    | 0.718  | 1.030  | .697    |
| 5X         | 42.5 | 42.5 | 1.498| 1.394         | 1.264          | 1.813| 7.57 | 3.350         | 0.76    | 0.481  | 0.575  | .837    |
| 6-5X       | 138.5| 42.5 | 1.499| 1.398         | 1.462          | 1.813| 1.320| 3.000         | 4.30    | 0.881  | 1.000  | .881    |
| 5X         | 90.5 | 42.5 | 1.500| 1.406         | 1.443          | 1.813| 1.336| 3.110         | 2.84    | 0.914  | 1.000  | .914    |
| 5X         | 42.5 | 42.5 | 1.498| 1.282         | 0.472          | 1.813| 1.285| 3.460         | 1.28    | 0.952  | 1.005  | .947    |
| 7-5Y       | 138.5| 42.5 | 1.609| 1.520         | 1.436          | 1.813| 0.776| 3.020         | 2.53    | 1.021  | 1.004  | 1.017   |
| 5Y         | 90.5 | 42.5 | 1.607| 1.534         | 1.476          | 1.813| 0.776| 2.560         | 1.65    | 0.880  | 1.030  | .854    |
| 5Y         | 42.5 | 42.5 | 1.606| 1.533         | 1.423          | 1.813| 0.819| 1.740         | 0.82    | 0.675  | 0.635  | 1.063   |
| 8-5X       | 138.5| 42.5 | 1.610| 1.551         | 1.492          | 1.813| 1.336| 1.590         | 4.36    | 0.942  | 1.000  | .942    |
| 9-5X       | 138.5| 42.5 | 1.743| 1.751         | 1.100          | 1.813| 0.910| 1.265         | 2.97    | 0.957  | 1.000  | .957    |
| 10-5X      | 138.5| 42.5 | 1.748| 1.208         | 0.753          | 1.813| 1.126| 1.015         | 3.67    | 1.017  | 1.000  | 1.017   |
| 11-Y       | 138.5| 42.5 | 1.995| 0.825         | 0.900          | 1.805| 0.979| 0.494         | 3.19    | 0.908  | 1.000  | .908    |
| 12-7X      | 138.5| 42.5 | 1.001| 0.988         | 0.899          | 1.805| 1.028| 0.487         | 3.35    | 1.050  | 1.000  | 1.050   |

Mean................................................. | .921 |

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Table 29.—The ratios $E_a/E_L$ and $E_b/E_L$ for spruce plywood. Plies of equal thickness

<table>
<thead>
<tr>
<th>Plies</th>
<th>$E_a/E_L$</th>
<th>$E_b/E_L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.679</td>
<td>0.357</td>
</tr>
<tr>
<td>5</td>
<td>0.614</td>
<td>0.422</td>
</tr>
<tr>
<td>7</td>
<td>0.587</td>
<td>0.449</td>
</tr>
<tr>
<td>9</td>
<td>0.572</td>
<td>0.464</td>
</tr>
</tbody>
</table>

Table 30.—The ratios $E_a/E_L$ and $E_b/E_L$ for Douglas-fir plywood. Plies of equal thickness

<table>
<thead>
<tr>
<th>Plies</th>
<th>$E_a/E_L$</th>
<th>$E_b/E_L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.686</td>
<td>0.372</td>
</tr>
<tr>
<td>5</td>
<td>0.623</td>
<td>0.434</td>
</tr>
<tr>
<td>7</td>
<td>0.596</td>
<td>0.461</td>
</tr>
<tr>
<td>9</td>
<td>0.581</td>
<td>0.476</td>
</tr>
</tbody>
</table>

Table 31.—Plate of type 3Y

<table>
<thead>
<tr>
<th>$k$ :</th>
<th>$t$ :</th>
<th>$H$ :</th>
<th>$Q$ :</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0  :</td>
<td>(See table 32.)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.5  :</td>
<td>1.50</td>
<td>1.7390</td>
<td>5.862</td>
</tr>
<tr>
<td>2.0  :</td>
<td>2.00</td>
<td>1.9005</td>
<td>4.729</td>
</tr>
<tr>
<td>3.0  :</td>
<td>3.00</td>
<td>5.277</td>
<td>4.279</td>
</tr>
<tr>
<td>4.0  :</td>
<td>3.50</td>
<td>4.706</td>
<td>4.720</td>
</tr>
<tr>
<td>5.0  :</td>
<td>3.51</td>
<td>4.685</td>
<td>5.312</td>
</tr>
<tr>
<td>$\infty$ :</td>
<td>$\ldots$</td>
<td>4.637</td>
<td>6.897</td>
</tr>
</tbody>
</table>
Table 32.—Plate of type 3X

<table>
<thead>
<tr>
<th>k</th>
<th>ς</th>
<th>H</th>
<th>Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.953</td>
<td>6.327</td>
<td>13.30</td>
</tr>
<tr>
<td>2.0</td>
<td>0.987</td>
<td>6.236</td>
<td>13.11</td>
</tr>
<tr>
<td>3.0</td>
<td>0.969</td>
<td>6.226</td>
<td>13.11</td>
</tr>
<tr>
<td>∞</td>
<td></td>
<td>6.208</td>
<td>13.12</td>
</tr>
</tbody>
</table>

Table 33.—Plate of type 5Y

<table>
<thead>
<tr>
<th>k</th>
<th>ς</th>
<th>H</th>
<th>Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>1.50</td>
<td>2.408</td>
<td>6.465</td>
</tr>
<tr>
<td>1.5</td>
<td>1.50</td>
<td>2.408</td>
<td>6.465</td>
</tr>
<tr>
<td>2.0</td>
<td>2.00</td>
<td>1.679</td>
<td>5.434</td>
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Table 34.—Plate of type 5X

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Table 36.—Plate of type 3Y

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### Table 39.---Plate of type 5X

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### Table 40.---Isotropic plate ( $\sigma = 0.25$ )

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1In the calculations the elastic constants of spruce were used. The values as written indicate an accuracy to a greater number of decimal places than one is warranted in assuming from the nature of the calculations.
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**Table 43.—Plate of type 3v. Uniform load. Edges simply supported. Exact and approximate methods**

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<td>Lb. per sq. ft.</td>
<td>Lb. per sq. in.</td>
<td>Lb. per sq. in.</td>
<td>Lb. per sq. in.</td>
<td>Lb. per sq. in.</td>
<td>Lb. per sq. in.</td>
<td>Lb. per sq. in.</td>
</tr>
<tr>
<td>1</td>
<td>0.4638</td>
<td>1.001</td>
<td>1.0010</td>
<td>83.7</td>
<td>81.8</td>
<td>67.4</td>
<td>66.6</td>
</tr>
<tr>
<td>6</td>
<td>0.8992</td>
<td>6.089</td>
<td>1.0150</td>
<td>151.7</td>
<td>158.5</td>
<td>249.6</td>
<td>250.2</td>
</tr>
<tr>
<td>20</td>
<td>1.3410</td>
<td>19.400</td>
<td>0.9700</td>
<td>219.7</td>
<td>236.4</td>
<td>578.0</td>
<td>556.4</td>
</tr>
<tr>
<td>50</td>
<td>1.8270</td>
<td>48.290</td>
<td>0.9658</td>
<td>296.7</td>
<td>322.1</td>
<td>1073.0</td>
<td>1032.8</td>
</tr>
<tr>
<td>200</td>
<td>2.9180</td>
<td>194.600</td>
<td>0.9730</td>
<td>474.1</td>
<td>514.5</td>
<td>2707.0</td>
<td>2634.7</td>
</tr>
</tbody>
</table>

**Table 44.—Plate of type 3x. Uniform load. Edges simply supported. Exact and approximate methods**

<table>
<thead>
<tr>
<th>q</th>
<th>( \frac{w_c}{h} )</th>
<th>q_a</th>
<th>q_a</th>
<th>s_1</th>
<th>s_1a</th>
<th>g_1</th>
<th>g_1a</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lb. per sq. ft.</td>
<td>Lb. per sq. ft.</td>
<td>Lb. per sq. in.</td>
<td>Lb. per sq. in.</td>
<td>Lb. per sq. in.</td>
<td>Lb. per sq. in.</td>
<td>Lb. per sq. in.</td>
<td>Lb. per sq. in.</td>
</tr>
<tr>
<td>4</td>
<td>0.4360</td>
<td>4.066</td>
<td>1.0160</td>
<td>249.2</td>
<td>230.6</td>
<td>57.1</td>
<td>58.8</td>
</tr>
<tr>
<td>20</td>
<td>0.9797</td>
<td>20.270</td>
<td>1.0140</td>
<td>541.8</td>
<td>518.2</td>
<td>294.7</td>
<td>297.0</td>
</tr>
<tr>
<td>65</td>
<td>1.5600</td>
<td>66.210</td>
<td>1.0190</td>
<td>830.6</td>
<td>825.2</td>
<td>746.1</td>
<td>753.1</td>
</tr>
<tr>
<td>175</td>
<td>2.2240</td>
<td>176.900</td>
<td>1.0110</td>
<td>1140.0</td>
<td>1176.0</td>
<td>1527.0</td>
<td>1530.0</td>
</tr>
<tr>
<td>500</td>
<td>3.1850</td>
<td>497.700</td>
<td>0.9954</td>
<td>1593.0</td>
<td>1685.0</td>
<td>3160.0</td>
<td>3140.0</td>
</tr>
</tbody>
</table>
Table 45.—Plate of type 5Y. Uniform load. Edges simply supported.  
Exact and approximate methods

<table>
<thead>
<tr>
<th>q : ( \frac{w_0}{h} )</th>
<th>q : ( \frac{q}{a} )</th>
<th>q : ( \frac{q}{a} )</th>
<th>s_2</th>
<th>s_2a</th>
<th>( \varepsilon_2 )</th>
<th>( \varepsilon_{2a} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lb. per sq. ft. :</td>
<td>Lb. per sq. ft. :</td>
<td>Lb. per sq. in. :</td>
<td>Lb. per sq. in. :</td>
<td>Lb. per sq. in. :</td>
<td>Lb. per sq. in. :</td>
<td></td>
</tr>
<tr>
<td>8 : 0.3648 : 7.928 : 0.9910 : 344.9 : 321.5 : 118.7 : 114.4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>50 : .8281 : 50.380 : 1.0080 : 747.8 : 729.9 : 591.1 : 589.4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>250 : 1.4770 : 246.200 : .9848 : 1261.0 : 1302.0 : 1919.0 : 1875.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>750 : 2.1350 : 715.000 : .9533 : 1770.0 : 1882.0 : 4108.0 : 3919.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 46.—Plate of type 5X. Uniform load. Edges simply supported.  
Exact and approximate methods

<table>
<thead>
<tr>
<th>q : ( \frac{w_0}{h} )</th>
<th>q : ( \frac{q}{a} )</th>
<th>q : ( \frac{q}{a} )</th>
<th>s_1</th>
<th>s_1a</th>
<th>( \varepsilon_1 )</th>
<th>( \varepsilon_{1a} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lb. per sq. ft. :</td>
<td>Lb. per sq. ft. :</td>
<td>Lb. per sq. in. :</td>
<td>Lb. per sq. in. :</td>
<td>Lb. per sq. in. :</td>
<td>Lb. per sq. in. :</td>
<td></td>
</tr>
<tr>
<td>25 : 0.4231 : 25.45 : 1.0180 : 671.1 : 621.7 : 148.9 : 153.9</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>150 : 1.0190 : 151.60 : 1.0110 : 1557.0 : 1497.3 : 890.5 : 892.6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>500 : 1.6240 : 509.40 : 1.0190 : 2384.0 : 2386.0 : 2243.0 : 2268.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1250 : 2.2370 : 1248.00 : .9984 : 3171.0 : 3287.0 : 4348.0 : 4301.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 47.—Steel plate. Uniform load. Edges simply supported.  
Exact and approximate methods

<table>
<thead>
<tr>
<th>q : $\frac{v_0}{h}$</th>
<th>$q_a$</th>
<th>$\frac{q_a}{q}$</th>
<th>s</th>
<th>$s_a$</th>
<th>g</th>
<th>$g_a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lb. per sq. ft.</td>
<td>: Lb. per sq. ft.</td>
<td>: Lb. per sq. in.</td>
<td>Lb. per sq. in.</td>
<td>Lb. per sq. in.</td>
<td>Lb. per sq. in.</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>0.5009</td>
<td>102.0 : 1.020</td>
<td>4,748.0 : 4,435.0 : 1,261.0 : 1,299.0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>500</td>
<td>1.0280</td>
<td>512.0 : 1.024</td>
<td>9,301.0 : 9,101.0 : 5,410.0 : 5,473.0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1000</td>
<td>1.3340</td>
<td>1,015.0 : 1.015</td>
<td>11,367.0 : 11,811.0 : 9,157.0 : 9,216.0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2000</td>
<td>1.7200</td>
<td>2,048.0 : 1.024</td>
<td>14,865.0 : 15,228.0 : 15,133.0 : 15,321.0</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 48.—Plate of type 3Y. Uniform load. Clamped edges.  
Exact and approximate methods

<table>
<thead>
<tr>
<th>q : $\frac{w_0}{h}$</th>
<th>$q_a$</th>
<th>$\frac{q_a}{q}$</th>
<th>s_2</th>
<th>s_2a</th>
<th>g_2</th>
<th>g_2a</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lb. per sq. ft.</td>
<td>: Lb. per sq. ft.</td>
<td>: Lb. per sq. in.</td>
<td>Lb. per sq. in.</td>
<td>Lb. per sq. in.</td>
<td>Lb. per sq. in.</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.4614</td>
<td>1.973 : 0.9865</td>
<td>366.7 : 354.0 : 63.0 : 63.5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.7780</td>
<td>5.967 : 0.9945</td>
<td>796.4 : 777.8 : 178.4 : 130.6</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.9562</td>
<td>9.890 : 0.9890</td>
<td>1122.0 : 1130.9 : 273.5 : 272.8</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>1.1240</td>
<td>15.020 : 1.0010</td>
<td>1481.0 : 1561.2 : 375.6 : 377.0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>1.3550</td>
<td>24.820 : 0.9928</td>
<td>2093.0 : 2342.4 : 556.2 : 547.8</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 49.—Plate of type 3X. Uniform load. Clamped edges. 
Exact and approximate methods

<table>
<thead>
<tr>
<th>q</th>
<th>w₀/h</th>
<th>qₐ</th>
<th>qₐ/q</th>
<th>s₁</th>
<th>s₁a</th>
<th>ε₁</th>
<th>ε₁a</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lb. per sq. ft.</td>
<td>Lb. per sq. ft.</td>
<td>Lb. per sq. in.</td>
<td>Lb. per sq. in.</td>
<td>Lb. per sq. in.</td>
<td>Lb. per sq. in.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>0.4209</td>
<td>14.95</td>
<td>0.9967</td>
<td>836.6</td>
<td>828.4</td>
<td>52.2</td>
<td>52.9</td>
</tr>
<tr>
<td>40</td>
<td>0.8809</td>
<td>39.94</td>
<td>0.9985</td>
<td>1897.0</td>
<td>1866.7</td>
<td>227.3</td>
<td>231.6</td>
</tr>
<tr>
<td>100</td>
<td>1.4610</td>
<td>98.85</td>
<td>0.9885</td>
<td>3734.0</td>
<td>3596.1</td>
<td>631.3</td>
<td>637.1</td>
</tr>
<tr>
<td>250</td>
<td>2.2170</td>
<td>251.30</td>
<td>1.0050</td>
<td>7034.0</td>
<td>7010.0</td>
<td>1435.0</td>
<td>1467.0</td>
</tr>
</tbody>
</table>

Table 50.—Plate of type 5Y. Uniform load. Clamped edges. 
Exact and approximate methods

<table>
<thead>
<tr>
<th>q</th>
<th>w₀/h</th>
<th>qₐ</th>
<th>qₐ/q</th>
<th>s₂</th>
<th>s₂a</th>
<th>ε₂</th>
<th>ε₂a</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lb. per sq. ft.</td>
<td>Lb. per sq. ft.</td>
<td>Lb. per sq. in.</td>
<td>Lb. per sq. in.</td>
<td>Lb. per sq. in.</td>
<td>Lb. per sq. in.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>0.4022</td>
<td>29.89</td>
<td>0.9963</td>
<td>1377.0</td>
<td>1358.4</td>
<td>131.6</td>
<td>134.1</td>
</tr>
<tr>
<td>50</td>
<td>0.5670</td>
<td>49.26</td>
<td>0.9852</td>
<td>2069.0</td>
<td>2011.5</td>
<td>266.6</td>
<td>266.4</td>
</tr>
<tr>
<td>150</td>
<td>1.0230</td>
<td>147.30</td>
<td>0.9820</td>
<td>4592.0</td>
<td>4419.8</td>
<td>871.7</td>
<td>867.4</td>
</tr>
<tr>
<td>300</td>
<td>1.3920</td>
<td>298.10</td>
<td>0.9937</td>
<td>7403.0</td>
<td>7336.0</td>
<td>1582.0</td>
<td>1606.0</td>
</tr>
</tbody>
</table>
Table 51. Plate of type 5X. Uniform load. Clamped edges. Exact and approximate methods

<table>
<thead>
<tr>
<th>q</th>
<th>( \frac{w_0}{h} )</th>
<th>q_a</th>
<th>( \frac{q_a}{q} )</th>
<th>s_1</th>
<th>s_{1a}</th>
<th>( \epsilon_1 )</th>
<th>( \epsilon_{1a} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lb. per</td>
<td>Lb. per</td>
<td>Lb. per</td>
<td>Lb. per</td>
<td>Lb. per</td>
<td>Lb. per</td>
<td>Lb. per</td>
<td>Lb. per</td>
</tr>
<tr>
<td>sq. ft.</td>
<td>sq. ft.</td>
<td>sq. in.</td>
<td>sq. in.</td>
<td>sq. in.</td>
<td>sq. in.</td>
<td>sq. in.</td>
<td>sq. in.</td>
</tr>
<tr>
<td>100</td>
<td>0.4329</td>
<td>99.55</td>
<td>0.9955</td>
<td>2399.0</td>
<td>2375.1</td>
<td>153.6</td>
<td>155.4</td>
</tr>
<tr>
<td>200</td>
<td>.7366</td>
<td>199.40</td>
<td>0.9970</td>
<td>4290.0</td>
<td>4241.4</td>
<td>440.9</td>
<td>449.8</td>
</tr>
<tr>
<td>350</td>
<td>1.0440</td>
<td>348.20</td>
<td>0.9949</td>
<td>6621.0</td>
<td>6448.6</td>
<td>886.3</td>
<td>903.5</td>
</tr>
<tr>
<td>600</td>
<td>1.3750</td>
<td>585.00</td>
<td>0.9750</td>
<td>9736.0</td>
<td>9335.0</td>
<td>1596.0</td>
<td>1567.0</td>
</tr>
</tbody>
</table>

Table 52. Steel plate. Uniform load. Clamped edges. Exact and approximate methods

<table>
<thead>
<tr>
<th>q</th>
<th>( \frac{w_0}{h} )</th>
<th>q_a</th>
<th>( \frac{q_a}{q} )</th>
<th>s</th>
<th>s_a</th>
<th>( \epsilon )</th>
<th>( \epsilon_a )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lb. per</td>
<td>Lb. per</td>
<td>Lb. per</td>
<td>Lb. per</td>
<td>Lb. per</td>
<td>Lb. per</td>
<td>Lb. per</td>
<td>Lb. per</td>
</tr>
<tr>
<td>sq. ft.</td>
<td>sq. ft.</td>
<td>sq. in.</td>
<td>sq. in.</td>
<td>sq. in.</td>
<td>sq. in.</td>
<td>sq. in.</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>0.1726</td>
<td>96.8</td>
<td>0.9680</td>
<td>5,597.9</td>
<td>5,424.0</td>
<td>146.9</td>
<td>144.8</td>
</tr>
<tr>
<td>300</td>
<td>.4606</td>
<td>291.3</td>
<td>0.9710</td>
<td>15,462.0</td>
<td>14,963.0</td>
<td>1,015.0</td>
<td>1,031.0</td>
</tr>
<tr>
<td>500</td>
<td>.6664</td>
<td>497.0</td>
<td>0.9940</td>
<td>23,569.0</td>
<td>23,228.0</td>
<td>2,185.0</td>
<td>2,218.0</td>
</tr>
<tr>
<td>750</td>
<td>.8604</td>
<td>745.0</td>
<td>0.9933</td>
<td>32,320.0</td>
<td>31,518.0</td>
<td>3,616.0</td>
<td>3,697.0</td>
</tr>
<tr>
<td>1000</td>
<td>1.0606</td>
<td>982.0</td>
<td>0.9820</td>
<td>39,936.0</td>
<td>38,490.0</td>
<td>5,072.0</td>
<td>5,055.0</td>
</tr>
</tbody>
</table>
FIG. 1
CHOICE OF AXES FOR PLYWOOD PLATE

FIG. 2
MIDDLE PLANE OF PLYWOOD PLATE
**Fig. 3**
THE THREE PRINCIPAL DIRECTIONS IN WOOD

**Fig. 4**
SUBDIVISIONS OF A PLATE
IN THE APPROXIMATE METHOD

**Fig. 5**
CROSS SECTION OF ASSUMED FORM
OF THE DEFLECTED SURFACE OF A PLATE
IN THE APPROXIMATE METHOD
FIG. 10

FACTOR $\gamma$ [FORMULA (3.40)] AS A FUNCTION OF $\beta/a$, WHERE $\beta = b (E_1/E_2)^{1/3}$.

UNIFORM LOAD. EDGES SIMPLY SUPPORTED.
FIG. 11
FACTOR \( \tau \) [FORMULA (3.40)] AS A FUNCTION OF \( \beta/a \),
WHERE \( \beta = b(E_1/E_2) \), UNIFORM LOAD, EDGES SIMPLY SUPPORTED.
FIG. 12
CROSS-SECTIONAL VIEW OF
APPARATUS FOR UNIFORMLY LOADING A PLYWOOD PLATE,
EDGES SIMPLY SUPPORTED
FIG. 13
DEFLECTION ALONG CENTER LINE OF PANEL No.4-TYPE 3X
EDGES CLAMPED

FIG. 14
DEFLECTION ALONG CENTER LINE OF PANEL No.6-TYPE 5X
EDGES CLAMPED

FIG. 15
DEFLECTION ALONG CENTER LINE OF PANEL No.8-TYPE 5X
EDGES CLAMPED
FIG. 16
SUCCESSIVE DIAL READINGS FOR INCREASING LOADS
Panel No. 4 - Type 3x4x8'

FIG. 17
SUCCESSIVE DIAL READINGS FOR INCREASING LOADS
Panel No. 4 - Type 3x4x8'

Note: For comparison all dial readings are multiplied by a factor to give 0.100 on the dial at the center of the panel.
FIG. 18

FACTOR \( \gamma \) [FORMULA (4.35)] AS A FUNCTION OF \( \beta /a \),
WHERE \( \beta = b \left( E_1 / E_2 \right)^{1/2} \). UNIFORM LOAD. EDGES CLAMPED.
FIG. 19
FACTOR $\gamma$ [FORMULA (4.35)] AS A FUNCTION OF $\beta/\alpha$, WHERE $\beta = b (E_1/E_2)^{\frac{1}{2}}$. UNIFORM LOAD. EDGES CLAMPED.
Fig. 20

Deflection and bending moments along the line $x=0.5a$ and $y=0$ of an infinite strip of plywood of type 3X with a concentrated load at the point $(0.5a, 0)$. 

Scale of $y$

Scale of $y$

Scale of $m_yP$ and $m_xP$

Scale of $w/(Pa^2/D)$
FIG. 21
INFINITE STRIP WITH A LOAD ACTING OVER A SMALL RECTANGULAR AREA.

FIG. 22
DEFLECTION AND BENDING MOMENTS ALONG THE LINES X = 0.5a AND Y = 0 OF AN INFINITE STRIP OF PLYWOOD OF TYPE 3X WITH A LOAD P UNIFORMLY DISTRIBUTED OVER A SMALL SQUARE WHOSE CENTER IS AT THE POINT (0.5a, 0).
\begin{align*}
\text{VI} & \quad y = 2b + y_i, \\
\text{V} & \quad y = 2b - y_i, \\
\text{IV} & \quad y = b \\
\text{III} & \quad y = -2b + y_i, \\
\text{II} & \quad y = -2b \\
\text{I} & \quad y = y_i \\
\end{align*}

\text{FIG. 23}

PLYWOOD PLATE UNDER A LOAD CONCENTRATED
AT A POINT AND THE CORRESPONDING DISTRIBUTION
OF LOADS ON AN INFINITE STRIP.

\text{FIG. 24}

DEFLECTION AND BENDING MOMENTS ALONG THE
LINES \(x = 0.5a\) AND \(y = 0\) OF A SQUARE PLATE OF
PLYWOOD OF TYPE 3X WITH A CONCENTRATED
LOAD AT THE POINT \((0.5a, 0)\).
FIG. 25

DEFLECTION AND BENDING MOMENTS ALONG THE LINES $x=0.5a$ AND $y=0$ OF A SQUARE PLATE OF PLYWOOD OF TYPE 3X WITH A LOAD $P$ UNIFORMLY DISTRIBUTED OVER A SMALL SQUARE WHOSE CENTER IS AT THE POINT $(0.5a, 0)$.  

Z M 39974 P
FIG. 26
FACTOR $\gamma$ \[\text{FORMULA (7.5)}\] AS A FUNCTION OF $\beta/\alpha$, WHERE $\beta = b \left(\frac{E_1}{E_2}\right)^{1/2}$. CONCENTRATED LOAD, EDGES SIMPLY SUPPORTED.

$2 \times 39975 f$
FIG. 27
THE COEFFICIENT $\alpha$ IN THE FORMULA (9.3)
AS A FUNCTION OF $\eta$

FIG. 28
THE COEFFICIENT $\alpha$ IN THE FORMULA (10.16)
AS A FUNCTION OF $\eta$
FIG. 29
MOMENTS AND SHEARING FORCES
ACTING ON AN ELEMENT OF A PLATE
FIG. 30
INFINITE STRIP OF PLYWOOD UNDER A LOAD ACTING OVER A SEGMENT OF A LINE

FIG. 31
PLYWOOD PLATE UNDER A LOAD CONCENTRATED AT A POINT ON ITS CENTRAL LINE