BUCKLING OF SANDWICH CYLINDERS OF FINITE LENGTH UNDER UNIFORM EXTERNAL LATERAL PRESSURE

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In Cooperation with the University of Wisconsin
BUCKLING OF SANDWICH CYLINDERS OF FINITE LENGTH

UNDER UNIFORM EXTERNAL LATERAL PRESSURE*

By

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Summary

A theoretical analysis is made of the problem of the buckling of circular cylinders of sandwich construction acted upon by uniform external lateral pressure. The solution obtained is based on the assumption that the sandwich cylinder is comprised of isotropic, membrane facings and an orthotropic core. The mathematical solution of the problem, which is in the form of a characteristic determinant of sixth order, is applicable to sandwich cylinders of any length and of any core thickness. Numerical results are obtained for various values of the parameters that enter the problem, and curves are included which illustrate how the critical load varies as the values of these parameters are varied.

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Introduction

Sandwich construction, as it is usually employed, consists of two relatively thin sheets of a material that has comparatively high strength and stiffness properties separated by and bonded to a relatively thick layer of a lightweight material that has comparatively low strength and stiffness properties. The two outer sheets are commonly called "facings," and the inner layer is commonly called the "core." Sandwich construction results in a composite structural element having a higher strength-weight ratio than can be obtained through the use of a single homogeneous material. For this reason, its major fields of application are in structures in which weight is a prime factor, such as aircraft and guided missiles. However, the development of improved adhesives and the improvement of methods of fabrication are making its use practical in an increasingly large number of different types of structures.

With the increased use of sandwich construction has come an increased need for reliable design data. Much work of both a theoretical and an experimental nature has been done in an effort to provide this needed information. The physical properties of many of the different materials used in sandwich construction have been determined, and analyses of a number of stress and stability problems have been performed. Progress has been somewhat retarded because of the many variables that must be taken into account in sandwich analysis and the fact that the analysis of a layered system is inherently a difficult problem. A rather complete bibliography of the part of this work that pertains particularly to shells and shell-like structures is contained in a recent U. S. Navy publication (1).
The purpose of this thesis is to obtain the solution for the critical load on circular sandwich cylinders of finite length subjected to uniform external lateral pressure. The critical load is defined as the intensity of pressure at which the cylinder buckles due to lack of stiffness. This particular problem was chosen because it represents a fundamental problem in sandwich analysis that has not previously been solved. The need for such an analysis has arisen in connection with the design of certain component parts of aircraft and guided missiles. It is believed that the method of analysis used here is somewhat more rigorous than methods that have been used to obtain solutions to other problems in sandwich analysis. For this reason, the general method of approach may be of use in obtaining solutions to other new problems as well as in obtaining better solutions to some problems that have already been solved. The solution contained in this thesis represents an extension of a previous solution of the problem of the buckling of long sandwich cylinders (2).

In this thesis it is assumed that the sandwich cylinder is comprised of very thin, isotropic facings of a relatively stiff material separated by an orthotropic core. The analysis of the facings is based on membrane theory.* The core is considered to have such a low load-carrying capacity in the tangential and longitudinal directions as compared to the facings that the normal stress in the core in these directions and the shear stress in the core on planes perpendicular to the facings and in these directions may be neglected. This core assumption

*An analysis taking into account the stiffnesses of the individual facings may be made along the lines illustrated in Ref. 3.
has been widely used in sandwich analysis and is known to represent actual sandwich construction very well. The fact that no further simplifying assumptions in regard to the core are needed, other than the assumption that it behaves as an elastic continuum, is emphasized. It is felt that some analyses of sandwich construction are not sufficiently accurate in certain ranges of physical properties and dimensions because of additional simplifying assumptions made in regard to the core. The action of the core and facings is related by the assumption that their displacements are equal at the interfaces between the core and facings. These interfaces are assumed to be at the middle surfaces of the facings.

The method used for establishing the stability criterion is similar in concept to that used by Timoshenko in the analysis of the buckling of homogeneous cylinders of finite length subjected to uniform external lateral pressure.* The assumption is made that, for pressures less than the critical, the circular cylinder remains circular and the only stresses present are a uniform circumferential stress in the facings and a uniform radial stress in the core. In discussing the buckling of the sandwich cylinder, only small deflections from this uniformly compressed form of equilibrium are considered; thus, the stresses induced in the cylinder as it goes from the circular to the slightly deformed configuration may be considered small as compared to the pre-buckling stresses.

*See Ref. 4, Art. 83.
### Notation

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$, $\theta$, $z$</td>
<td>radial, tangential, and longitudinal coordinates, respectively</td>
</tr>
<tr>
<td>$a$</td>
<td>radius to middle surface of outer facing</td>
</tr>
<tr>
<td>$b$</td>
<td>radius to middle surface of inner facing</td>
</tr>
<tr>
<td>$t$</td>
<td>thickness of each facing</td>
</tr>
<tr>
<td>$l$</td>
<td>length of cylinder</td>
</tr>
<tr>
<td>$E$</td>
<td>modulus of elasticity of facings</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Poisson's ratio of facings</td>
</tr>
<tr>
<td>$G$</td>
<td>modulus of rigidity of facings</td>
</tr>
<tr>
<td>$E_c$</td>
<td>modulus of elasticity of core in direction normal to facings</td>
</tr>
<tr>
<td>$G_{r\theta}$</td>
<td>modulus of rigidity of core in $r\theta$ plane</td>
</tr>
<tr>
<td>$G_{rz}$</td>
<td>modulus of rigidity of core in $rz$ plane</td>
</tr>
<tr>
<td>$q$</td>
<td>intensity of uniform external lateral loading</td>
</tr>
<tr>
<td>$k$</td>
<td>[ \frac{1}{l + \frac{b}{a} - \frac{Et \log b}{E_c a}} ]</td>
</tr>
<tr>
<td>$\sigma_r$</td>
<td>small normal stress in core in radial direction</td>
</tr>
<tr>
<td>$\tau_{r\theta}$, $\tau_{rz}$</td>
<td>small transverse shear stresses in core</td>
</tr>
<tr>
<td>$u_c$, $v_c$, $w_c$</td>
<td>small radial, tangential, and longitudinal core displacements, respectively</td>
</tr>
<tr>
<td>$\epsilon_r$, $\epsilon_\theta$, $\epsilon_z$</td>
<td>small radial, tangential, and longitudinal core normal strains, respectively</td>
</tr>
<tr>
<td>$\gamma_{r\theta}$, $\gamma_{rz}$, $\gamma_{\theta z}$</td>
<td>small radial, tangential, and longitudinal core shearing strains, respectively</td>
</tr>
<tr>
<td>$n$</td>
<td>number of waves in circumference of buckled cylinder</td>
</tr>
</tbody>
</table>
\[ \lambda = \frac{\pi a}{l} \]

\[ \varepsilon_{n\theta} = \frac{E_c}{2G_{r\theta}} - \frac{n^2}{r^2} \]

\[ \varepsilon_z = \frac{E_c}{G_{r\theta}} \]

\[ \varepsilon', \varepsilon' \]

small normal strains in tangential direction in outer and inner facings, respectively

\[ \varepsilon', \varepsilon' \]

small normal strains in longitudinal direction in outer and inner facings, respectively

\[ u, v, w \]

small radial, tangential, and longitudinal displacements of outer facing

\[ u', v', w' \]

small radial, tangential, and longitudinal displacements of inner facing

\[ N_{\theta}, N_z, N_{\theta z} \]

normal forces and shear force per unit length of outer facing

\[ N', N', N' \]

normal forces and shear force per unit length of inner facing

\[ \beta = \frac{E a (1 - \mu^2)}{E_c} \]

\[ \alpha = \frac{qa (1 - \mu^2)}{E_t} \]

log

natural logarithm

\[ A_n, B_n, C_n, D_n, L_n, R_n \]

arbitrary constants
In the analysis of the sandwich cylinder, cylindrical coordinates \( r, \theta, \) and \( z \) are used; the dimensions of the cylinder and the positive directions of the coordinates are indicated in figure 1. The radii to the middle surfaces of the outer and inner facings are denoted by \( a \) and \( b \), respectively, and the thickness of each facing is denoted by \( t \). The origin of the coordinate system is placed at the middle cross-section of the cylinder whose length is denoted by \( l \). Since \( q \) is considered to be positive when it acts in the positive \( r \)-direction, buckling occurs at a negative value of \( q \).

As mentioned previously, the cylinder at the instant before buckling is assumed to be in a state of uniform compression with the same stress condition existing in each cross-section. This assumption amounts to saying that the ends of the cylinder are not supported until after the cylinder has been initially compressed by a uniform load just less than the critical load. The expressions for the pre-buckling stresses are known,* and they are shown in figure 2. The radial stress in the core is equal to \( q \frac{a}{b} k \) and the circumferential force per unit length in the outer and inner facing is \( qa(1 - k) \) and \( qak \), respectively, where \( q \) is the intensity of uniform external lateral loading in the positive \( r \) direction and

\[
k = \frac{1}{l + \frac{b}{a} \left( \frac{E_t \log \frac{b}{a}}{E_c a} \right)}
\]

*See Ref. 2.
In discussing the equilibrium of the cylinder after it has buckled into a slightly deformed shape, it is assumed that the existing stress condition differs only slightly from the stress condition shown in figure 2. The analysis of the equilibrium of a differential element of the core of the deformed cylinder and of differential elements of the facings of the deformed cylinder is considered next.

Equilibrium of the Core

A differential element of the deformed core is shown in figure 3. The stresses, $\sigma_r$, $\tau_{r\theta}$, and $\tau_{rz}$, resulting from buckling are assumed to be small as compared to the pre-buckling stress, $q \frac{a}{r} k$. As indicated previously, the assumption is made that the core and facing materials are such that the stresses, $\sigma_\theta$, $\sigma_z$, and $\tau_{\theta z}$, in the core may be neglected.

The effect of the changes in geometry in the core was found to be negligible; therefore, for simplification, the equilibrium equations are written on the basis of the original geometry of the element. The summation of forces in the radial direction results in the following equilibrium equation:

$$
(- q \frac{a}{r} k - \sigma_r) r d\theta dz + (q \frac{a}{r} k + \sigma_r + \frac{d}{dr} q \frac{a}{r} k + \frac{\partial \sigma_r}{\partial r}) dr
dr dz + \tau_{r\theta} (r + dr) d\theta dr dz + \tau_{rz} (r + dr) dr d\theta dz = 0
$$

If terms containing the product of more than three differentials are neglected, the above equation may be written as

$$
\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r}{r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} = 0
$$

(1)
The summation of forces in the tangential direction yields
\[ - \tau_{r\theta} r \, d\theta \, dz + \left( \tau_{r\theta} + \frac{\partial \tau_{r\theta}}{\partial r} dr \right) (r + dr) \, d\theta \, dz + \tau_{r\theta} \, dr \, d\theta \, dz = 0 \]
or, neglecting differentials of higher order,
\[ \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2 \tau_{r\theta}}{r} = 0 \]
\[(2)\]
Finally, the summation of forces in the longitudinal direction yields
\[ - \tau_{rz} r \, d\theta \, dz + \left( \tau_{rz} + \frac{\partial \tau_{rz}}{\partial r} dr \right) (r + dr) \, d\theta \, dz = 0 \]
or
\[ \frac{\partial \tau_{rz}}{\partial r} + \frac{\tau_{rz}}{r} = 0 \]
\[(3)\]
If it is assumed that during buckling the generators of the cylinder deflect into a half-wave of a sine curve and the circumference subdivides into \( n \) waves, the core displacements take the following form:
\[ u_c = f_1(r) \cos n\theta \cos \frac{nz}{l} \]
\[ v_c = f_2(r) \sin n\theta \cos \frac{nz}{l} \]
\[ w_c = f_3(r) \cos n\theta \sin \frac{nz}{l} \]
\[(4)\]
where \( u_c \), \( v_c \), and \( w_c \) represent the small displacements of the core in the \( r \), \( \theta \), and \( z \) directions, respectively. At the ends of the cylinder \( u_c \) and \( \frac{\partial^2 u_c}{\partial z^2} \) are both zero, which represents the conditions of simply-
supported edges. The displacements $u$, $v$, and $w$ are related to the core strains by the following equations:

$$
\varepsilon_r = \frac{\partial u}{\partial r}; \quad \varepsilon_\theta = \frac{u}{r} + \frac{1}{r} \frac{\partial v}{\partial \theta}; \quad \varepsilon_z = \frac{\partial w}{\partial z}
$$

$$
\gamma_{r\theta} = \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}; \quad \gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}; \quad \gamma_{\theta z} = \frac{1}{r} \frac{\partial w}{\partial \theta} + \frac{\partial v}{\partial z}
$$

Also, since $\sigma_\theta$ and $\sigma_z$ are assumed to be zero,

$$
\sigma_r = E \varepsilon_r = Ec \frac{\partial u}{\partial r}
$$

in addition to

$$
\tau_{r\theta} = G_{r\theta} \gamma_{r\theta} = G_{r\theta} \left( \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \right)
$$

and

$$
\tau_{rz} = G_{rz} \gamma_{rz} = G_{rz} \left( \frac{\partial w}{\partial z} + \frac{\partial w}{\partial r} \right)
$$

If the expressions for the displacements given by equations (4) are substituted into equations (6), (7), and (8) and the resulting expressions for the stresses are required to satisfy the equilibrium equations (1),

*See Ref. 5, p. 305.*
(2), and (3), the functions of \( r \) that appear in equations (4) may be determined explicitly, thus enabling equations (4) to be written as follows:

\[
u_c = [A_n \frac{a^2}{r} + B_n a + C_n a \log \frac{r}{a} + D_n r] \cos n\theta \cos \frac{\pi z}{\ell}
\]

\[
u_c = \left[\frac{n}{n\theta} \frac{a^2}{r} - n B_n a - n C_n a (1 + \log \frac{r}{a}) + \right.
\]

\[
+ n D_n r \log \frac{r}{a} + \frac{L n r}{n} \sin n\theta \cos \frac{\pi z}{\ell}
\]

\[
w_c = [\lambda A_n a \log \frac{r}{a} + \lambda B_n r + \lambda C_n r (\log \frac{r}{a} - 1) + \lambda D_n a
\]

\[
+ \left(\frac{1}{2} \frac{r^2}{a^2} - \frac{\delta z}{\lambda^2} \log \frac{r}{a}\right) + R_n a] \cos n\theta \sin \frac{\pi z}{\ell}
\]

where \( \lambda = \frac{\pi a}{\ell} \)

\[
\delta n\theta = \frac{E_c}{2Gr\theta} - \frac{n^2}{2}
\]

\[
\delta z = \frac{E_c}{G_{rz}}
\]

and \( A_n, B_n, C_n, D_n, L_n, \) and \( R_n \) are arbitrary constants.

If equations (9), (10), and (11) are substituted into equations (6), (7), and (8), the expressions for the core stresses become

\[
\sigma_r = E_c \left(- A_n \frac{a^2}{r^2} + C_n \frac{a}{r} + D_n\right) \cos n\theta \cos \frac{\pi z}{\ell}
\]
It may be easily verified that equations (12), (13), and (14) satisfy the equilibrium equations, equations (1), (2), and (3). Furthermore, because of the manner in which they were derived, it may be stated that the right-hand sides of equations (9), (10), and (11) are unique functions insofar as their dependence upon $r$ is concerned.

**Equilibrium of the Facings**

Differential elements of the facings of the deformed cylinder are shown in figure 4. The quantities $N_\theta$, $N'_\theta$, $N_z$, $N'_z$, $N_\theta z$, and $N'_\theta z$ represent normal and shear forces per unit length of the facing upon which they act. The core is assumed to extend to the middle surfaces of the facings, and the stresses $(\sigma_r)_{r=a'}$, $(\sigma_r)_{r=b'}$, $(\tau_{r\theta})_{r=a'}$, $(\tau_{rz})_{r=a'}$, $(\tau_{r\theta})_{r=b'}$, and $(\tau_{rz})_{r=b'}$ are the stresses exerted by the core on the facings as a result of buckling. All of the above quantities are assumed to be small in comparison to the pre-buckling stresses shown in figure 4 as functions of the intensity of loading, $q$. The bending moments and transverse shear forces in the individual facings are neglected, in accordance with membrane theory. In formulating the equilibrium equations that apply to the facings, the effects of the rotation and stretching of the facings must be taken into account. Because of the stretching of the facings, the areas of the differential elements of the outer and
inner facings become \((1 + \varepsilon) (1 + \varepsilon_z) a \, d\theta \, dz\) and \((1 + \varepsilon') (1 + \varepsilon') \, b \, d\theta \, dz\), respectively. Since the middle surface strains are small quantities, their products may be neglected, and the differential areas of the outer and inner facings are expressed as \((1 + \varepsilon + \varepsilon_z) a \, d\theta \, dz\) and \((1 + \varepsilon' + \varepsilon') \, b \, d\theta \, dz\), respectively. The difference in the amount of rotation of the sides \(AB\) and \(CD\) with respect to the \(z\)-axis of the cylinder causes the central angle of the outer element to change from \(d\theta\) to \(\left(1 + \frac{1}{a} \frac{\partial y}{\partial \theta} - \frac{1}{a} \frac{\partial^2 u}{\partial \theta^2}\right) d\theta\). Similarly, the central angle of the inner element changes from \(d\theta\) to \(\left(1 + \frac{1}{b} \frac{\partial y'}{\partial \theta} - \frac{1}{b} \frac{\partial^2 u'}{\partial \theta^2}\right) d\theta\). Also, the difference in the amount of rotation of the sides \(AB\) and \(CD\) with respect to the \(r\)-axis is \(\left(\frac{\partial^2 y}{\partial \theta \partial z} + \frac{\partial u}{\partial z}\right)\); the corresponding difference in rotation of the sides \(A'B'\) and \(C'D'\) of the inner element is \(\left(\frac{\partial^2 y'}{\partial \theta \partial z} + \frac{\partial u'}{\partial z}\right)\). The foregoing expressions for the differences in rotation are as given by Timoshenko* and verified by Osgood and Joseph.** Since \(N_2\) and \(N_2'\) are small quantities, the relative rotation of the sides upon which they act may be neglected; the inclusion of this effect leads only to small terms of higher order.

The necessary changes in the geometry of the facings having been established, it is now possible to write three equilibrium equations for each facing. Considering first the differential element of the outer facing, the summation of forces in the direction perpendicular to the surface of the element results in the following equation:

*See Ref. 4, Art. 79.
**See Ref. 6.
- \( N_z (1 + \varepsilon_\theta) a \, d\theta + (N_z + \frac{\partial N_z}{\partial z}) (1 + \varepsilon_\theta) a \, d\theta - N_\theta z (1 + \varepsilon_z) \, dz \)

\[ + (N_\theta z + \frac{\partial N_\theta z}{\partial \theta}) (1 + \varepsilon_z) \, dz - (\tau_{rz})_{r=a} (1 + \varepsilon_\theta + \varepsilon_z) \]

\[ a \, d\theta \, dz - qa (1 - k) \left( \frac{\partial^2 v}{\partial \theta \partial z} + \frac{\partial u}{\partial z} \right) d\theta \, dz = 0 \]

If small quantities of higher order are again neglected, the above equation of equilibrium reduces to

\[ \frac{\partial N_z}{\partial z} + \frac{1}{a} \frac{\partial N_\theta z}{\partial \theta} = (\tau_{rz})_{r=a} + q (1 - k) \left( \frac{\partial^2 v}{\partial \theta \partial z} + \frac{\partial u}{\partial z} \right) \] (17)

The three equilibrium equations pertaining to the inner facing are obtained in a similar manner and may be written as follows:

\[ N'_\theta = b \left( \sigma_\theta \right)_{r=b} + q \frac{a}{b} k \left( u' + \frac{\partial^2 u'}{\partial \theta^2} \right) \] (18)

\[ \frac{1}{b} \frac{\partial N'_\theta}{\partial \theta} + \frac{\partial N'_z}{\partial z} = - (\tau_{r\theta})_{r=b} \] (19)

and

\[ \frac{\partial N'_z}{\partial z} + \frac{1}{b} \frac{\partial N'_\theta}{\partial \theta} = - (\tau_{rz})_{r=b} + q \frac{a}{b} k \left( \frac{\partial^2 v'}{\partial \theta \partial z} + \frac{\partial u'}{\partial z} \right) \] (20)

On the basis of Hooke's law, the following stress-strain relations in the facings are applicable:

\[ N_\theta = \frac{E_t}{1-\mu^2} (\varepsilon_\theta + \mu \varepsilon_z) \] (21)
\[ N_z = \frac{E_t}{1-\mu^2} (\varepsilon_z + \mu\varepsilon_\theta) \] (22)

\[ N_\theta z = G_t \gamma_\theta z = \frac{E_t}{2(1+\mu)} \gamma_\theta z \] (23)

\[ N'_\theta = \frac{E_t}{1-\mu^2} (\varepsilon'_\theta + \mu\varepsilon'_z) \] (24)

\[ N'_z = \frac{E_t}{1-\mu^2} (\varepsilon'_z + \mu\varepsilon'_\theta) \] (25)

and

\[ N'_\theta z = G_t \gamma'_\theta z = \frac{E_t}{2(1+\mu)} \gamma'_\theta z \] (26)

where \( E, G, \) and \( \mu \) represent the modulus of elasticity, the modulus of rigidity, and the Poisson's ratio of the facings, respectively. Since the strain-displacement relations given by equations (5) are applicable to the facings if \( r \) is replaced by \( a \) for the outer facing, \( r \) is replaced by \( b \) for the inner facing, and the corresponding displacements of the facings are substituted for the core displacements, equations (21) through (26) may be written as follows:

\[ N_\theta = \frac{E_t}{1-\mu^2} \left( \frac{u}{a} + \frac{1}{a} \frac{\partial v}{\partial \theta} + \mu \frac{\partial w}{\partial z} \right) \] (27)

\[ N_z = \frac{E_t}{1-\mu^2} \left( \mu \frac{u}{a} + \frac{\mu}{a} \frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} \right) \] (28)
If continuity of displacements at the cylinder interfaces is assumed, remembering that the core is considered to extend to the middle surfaces of the facings, then

\[ u = (u_c)_{r=a} \quad u' = (u_c)_{r=b} \]
\[ v = (v_c)_{r=a} \quad v' = (v_c)_{r=b} \]
\[ w = (w_c)_{r=a} \quad w' = (w_c)_{r=b} \]

The above relations enable equations (27) through (32) to be expressed as follows:

\[ N'_{\theta z} = \frac{Et}{2(1+\mu)} \left( \frac{\partial v}{\partial z} + \frac{1}{a} \frac{\partial w}{\partial \theta} \right) \]  
(29)

\[ N'_{\theta} = \frac{Et}{1-\mu^2} \left( \frac{u'}{b} + \frac{1}{b} \frac{\partial v'}{\partial \theta} + \mu \frac{\partial w'}{\partial z} \right) \]  
(30)

\[ N'_{z} = \frac{Et}{1-\mu^2} \left( \mu \frac{u'}{b} + \frac{\mu}{b} \frac{\partial v'}{\partial \theta} + \frac{\partial w'}{\partial z} \right) \]  
(31)

and

\[ N'_{\theta z} = \frac{Et}{2(1+\mu)} \left( \frac{\partial v'}{\partial z} + \frac{1}{b} \frac{\partial w'}{\partial \theta} \right) \]  
(32)

\[ N'_{\theta} = \frac{Et}{1-\mu^2} \left( \frac{u'}{b} + \frac{1}{b} \frac{\partial v'}{\partial \theta} + \mu \frac{\partial w'}{\partial z} \right) \]
Also, if account is taken of the continuity of displacements at the interfaces, the equilibrium equations of the facings, equations (15) through (20), may be written as follows:

\[ N_\theta = -a (\sigma_r)_{r=a} + q (1 - k) (u_c + \frac{\partial^2 u_c}{\partial \theta^2})_{r=a} \tag{39} \]

\[ \frac{1}{a} \frac{\partial N_\theta}{\partial \theta} + \frac{\partial N_\theta}{\partial z} = (\tau_{r\theta})_{r=a} \tag{40} \]

\[ \frac{\partial N_z}{\partial z} + \frac{1}{a} \frac{\partial N_\theta}{\partial \theta} = (\tau_{rz})_{r=a} + q (1 - k) \left( \frac{\partial^2 v_c}{\partial \theta \partial z} + \frac{\partial u_c}{\partial z} \right)_{r=a} \tag{41} \]
\[ N'_\theta = b \left( \sigma_r \right)_r=b + q \frac{a}{b} k \left( u_c + \frac{\partial^2 u_c}{\partial \theta^2} \right)_r=b \]  

(42)

\[ \frac{1}{b} \frac{\partial N'_\theta}{\partial \theta} + \frac{\partial N'_\theta}{\partial z} = - \left( \tau_{r\theta} \right)_r=b \]  

(43)

and

\[ \frac{\partial N'_z}{\partial z} + \frac{1}{b} \frac{\partial N'_z}{\partial \theta} = - \left( \tau_{rz} \right)_r=b + q \frac{a}{b} k \left( \frac{\partial^2 v}{\partial \theta \partial z} + \frac{\partial u_c}{\partial z} \right)_r=b \]  

(44)

If the expressions for the facing stresses given by equations (33) through (38) are substituted into the above equations and then a further substitution is made of the expressions for the core displacements and stresses given by equations (9) through (14), six equations containing the parameters \( A_n, B_n, C_n, D_n, L_n, \) and \( R_n \) that appear in the equations for the displacements of the core, equations (4), are obtained. These equations may be written as follows:
\[ \left[ 1 - \frac{\beta}{n^2} + \delta n \theta \left( 1 + \frac{1 - \mu}{2} \frac{\lambda^2}{n^2} \right) \right] A_n + \left[ - (n^2 - 1) + \mu \lambda^2 \right] B_n + \left[ - n^2 - \lambda^2 \right] C_n + \left[ 1 + \left( \frac{1 + \mu}{4} \right) \lambda^2 \right] D_n \]

\[ + [n + \left( \frac{1 - \mu}{2} \right) \frac{\lambda^2}{a^2} \] L_n + \left[ \left( \frac{1 + \mu}{2} \right) \lambda \right] R_n = 0 \] (45)

\[ \left[ 1 + \delta n \theta - \beta + (n^2 - 1) \alpha (1 - k) \right] A_n + \left[ - (n^2 - 1) + \mu \lambda^2 + (n^2 - 1) \alpha (1 - k) \right] B_n + \left[ - n^2 - \mu \lambda^2 \right] C_n + \left[ 1 + \frac{\mu \lambda^2}{2} + \beta + (n^2 - 1) \alpha (1 - k) \right] D_n + n L_n + \mu \lambda R_n = 0 \] (46)

\[ \left[ 1 + \frac{\beta}{n^2} \frac{b}{a} + \delta n \theta \left( 1 + \frac{1 - \mu}{2} \frac{\lambda^2}{n^2} \frac{b^2}{a^2} \right) + \left( \frac{1 + \mu}{2} \right) \lambda^2 \frac{b^2}{a^2} \log \frac{b}{a} \right] A_n + \left[ - (n^2 - 1) \frac{b}{a} + \mu \lambda^2 \frac{b^3}{a^3} \right] B_n + \left[ - n^2 \frac{b}{a} \right. \\

\left. - \lambda^2 \frac{b^3}{a^3} - (n^2 - 1) \frac{b}{a} \log \frac{b}{a} + \mu \lambda^2 \frac{b^3}{a^3} \log \frac{b}{a} \right] C_n + \left[ \frac{b^2}{a^2} + \left( \frac{1 + \mu}{4} \right) \lambda^2 \frac{b^4}{a^4} + n^2 \frac{b^2}{a^2} \log \frac{b}{a} \right. \\

\left. + \left( \frac{1 - \mu}{2} \right) \lambda^2 \frac{b^4}{a^4} \log \frac{b}{a} - \left( \frac{1 + \mu}{2} \right) \delta z \frac{b^2}{a^2} \log \frac{b}{a} \right] D_n + \left[ n \frac{b^2}{a^2} + \left( \frac{1 - \mu}{2} \right) \frac{\lambda^2}{n} \frac{b^4}{a^4} \right] L_n + \left[ \left( \frac{1 + \mu}{2} \right) \lambda \frac{b^2}{a^2} \right] R_n = 0 \] (47)
\[ [1 + \delta_{ng} + \beta \frac{b}{a} + \mu \lambda^2 \frac{b^2}{a^2} \log \frac{b}{a} + (n^2 - 1) \alpha \kappa] A_n + \left[ - (n^2 - 1) \frac{b}{a} + \mu \lambda^2 \frac{b^3}{a^3} + (n^2 - 1) \alpha \kappa \frac{b}{a} \right] B_n \]

\[ + \left[- n^2 \frac{b}{a} - (n^2 - 1) \frac{b}{a} \log \frac{b}{a} - \mu \lambda^2 \frac{b^3}{a^3} + \mu \lambda^2 \frac{b^3}{a^3} \log \frac{b}{a} - \beta \frac{b^2}{a^2} + (n^2 - 1) \alpha \kappa \frac{b}{a} \log \frac{b}{a} \right] C_n + \frac{b^2}{a^2} \]

\[ + n^2 \frac{b^2}{a^2} \log \frac{b}{a} + \mu \lambda^2 \frac{b^4}{a^4} - \mu \delta_z \frac{b^2}{a^2} \log \frac{b}{a} - \beta \frac{b^3}{a^3} + (n^2 - 1) \alpha \kappa \frac{b^2}{a^2} \right] D_n + n \frac{b^2}{a^2} L_n + \mu \lambda \frac{b^2}{a^2} R_n = 0 \] (48)

\[ \left[ \mu + \left( \frac{1 + \mu}{2} \right) \delta_{ng} - (\delta_{ng} + 1) \alpha (1 - k) \right] A_n + \left[ - \mu (n^2 - 1) + \lambda^2 + (n^2 - 1) \alpha (1 - k) \right] B_n + \left[ - n^2 - \lambda^2 \right. \]

\[ + n^2 \alpha (1 - k)] C_n + \left[ \mu + \left( \frac{1 - \mu}{4} \right) n^2 + \frac{\lambda^2}{\alpha^2} - \frac{b}{\lambda^2} - \alpha (1 - k) \right] D_n + \left( \frac{1 + \mu}{2} \right) n - n \alpha (1 - k) \right] L_n \]

\[ + \left[ \lambda + \left( \frac{1 - \mu}{2} \right) \frac{n^2}{\lambda} \right] R_n = 0 \] (49)
\[
\left[ \mu + \left( \frac{1+\mu}{2} \right) \delta n^2 + \left( \frac{1-\mu}{2} \right) \frac{n^2}{a^2} \log \frac{b}{a} + \frac{\lambda^2}{a^2} \frac{b^2}{a^3} \log \frac{b}{a} - (\delta n^2 + 1) \alpha k \right] A_n + \left[ -\mu \left( \frac{n^2}{a^2} - 1 \right) \frac{b}{a} + \frac{\lambda^2}{a^3} \frac{b^3}{a^3} \right] B_n + (n^2 - 1) \alpha k \frac{b}{a} C_n + \left[ \mu \frac{b^2}{a^2} + \left( \frac{1-\mu}{4} \right) \frac{n^2}{a^2} \frac{b^2}{a^2} + \frac{\lambda^2}{2} \frac{b^4}{a^4} + \left( \frac{1+\mu}{2} \right) \frac{n^2}{a^2} \frac{b^2}{a^3} \log \frac{b}{a} - \left( \frac{b^2}{a^2} \right) \right] D_n + \left[ \left( \frac{1-\mu}{2} \right) \frac{n^2}{a^2} - n \alpha k \frac{b^2}{a^2} \right] L_n + \left[ \frac{\lambda \frac{b^2}{a^2} + \left( \frac{1-\mu}{2} \right) \frac{n^2}{a^3}}{\lambda} \right] R_n = 0
\]

(50)

where \( \beta = \frac{E a (1-\mu^2)}{E t} \)

\( \alpha = \frac{q a (1-\mu^2)}{E t} \)

and, as defined previously,
Equations (45) through (50) are satisfied if the constants $A_n$, $B_n$, $C_n$, $D_n$, $L_n$, and $R_n$ are all equal to zero. This represents the uniformly compressed circular form of equilibrium of the cylinder. A buckled form of equilibrium is possible only if equations (45) through (50) yield non-zero solutions for the constants. This requires that the determinant of the coefficients of these constants be equal to zero. This determinant is shown on the following page.
| \( n \) | \( \frac{\phi(n)}{n} \) | \( \frac{\sum_{d|n} \phi(d)}{n} \) | \( \frac{\sum_{d|n} \phi_d(d)}{n} \) | \( \frac{\sum_{d|n} \phi_d(d)}{n} \) | \( \frac{\sum_{d|n} \phi_d(d)}{n} \) | \( \frac{\sum_{d|n} \phi_d(d)}{n} \) |
|-----|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 2   | \( \frac{1}{2} \) | \( \frac{1}{2} \) | \( \frac{1}{2} \) | \( \frac{1}{2} \) | \( \frac{1}{2} \) | \( \frac{1}{2} \) |
| 3   | \( \frac{2}{3} \) | \( \frac{2}{3} \) | \( \frac{2}{3} \) | \( \frac{2}{3} \) | \( \frac{2}{3} \) | \( \frac{2}{3} \) |
| 4   | \( \frac{3}{4} \) | \( \frac{3}{4} \) | \( \frac{3}{4} \) | \( \frac{3}{4} \) | \( \frac{3}{4} \) | \( \frac{3}{4} \) |
| 5   | \( \frac{4}{5} \) | \( \frac{4}{5} \) | \( \frac{4}{5} \) | \( \frac{4}{5} \) | \( \frac{4}{5} \) | \( \frac{4}{5} \) |
| 6   | \( \frac{5}{6} \) | \( \frac{5}{6} \) | \( \frac{5}{6} \) | \( \frac{5}{6} \) | \( \frac{5}{6} \) | \( \frac{5}{6} \) |
| 7   | \( \frac{6}{7} \) | \( \frac{6}{7} \) | \( \frac{6}{7} \) | \( \frac{6}{7} \) | \( \frac{6}{7} \) | \( \frac{6}{7} \) |
| 8   | \( \frac{7}{8} \) | \( \frac{7}{8} \) | \( \frac{7}{8} \) | \( \frac{7}{8} \) | \( \frac{7}{8} \) | \( \frac{7}{8} \) |
| 9   | \( \frac{8}{9} \) | \( \frac{8}{9} \) | \( \frac{8}{9} \) | \( \frac{8}{9} \) | \( \frac{8}{9} \) | \( \frac{8}{9} \) |
| 10  | \( \frac{9}{10} \) | \( \frac{9}{10} \) | \( \frac{9}{10} \) | \( \frac{9}{10} \) | \( \frac{9}{10} \) | \( \frac{9}{10} \) |
| 11  | \( \frac{10}{11} \) | \( \frac{10}{11} \) | \( \frac{10}{11} \) | \( \frac{10}{11} \) | \( \frac{10}{11} \) | \( \frac{10}{11} \) |
| 12  | \( \frac{11}{12} \) | \( \frac{11}{12} \) | \( \frac{11}{12} \) | \( \frac{11}{12} \) | \( \frac{11}{12} \) | \( \frac{11}{12} \) |
The lowest, real, negative value of \( \alpha \) for which this determinant equals zero is proportional to the critical load on the sandwich cylinder; this value of \( \alpha \) will be referred to hereafter as \( \alpha_{cr} \). After this value is determined, the value of the critical load is obtained from the definition of \( \alpha \) given previously, that is,

\[
\alpha_{cr} = \frac{q_{cr} \alpha (1-\mu^2)}{Et}
\]

Numerical Computations

Since the determinant in equation (51) contains the eigenvalue, \( \alpha \), in four of the six rows, the expansion of this determinant would result in a fourth order algebraic equation in \( \alpha \). Only the lowest of the four roots of this equation, the root that corresponds to the type of buckling illustrated in figure 5, is considered. Actually, for very short cylinders the root that corresponds to a face wrinkling type of buckling may become the lowest; a satisfactory analysis of this type of buckling requires that the stiffnesses of the individual facings be taken into account. Since this analysis is based on the assumption of membrane facings, no attempt is made to determine the critical loads for this type of buckling.

A literal expansion of the determinant in equation (51) was found to be impractical. It was decided that numerical solutions obtained by the use of machine methods would be the most practical way to obtain design data.* Originally, the sixth order determinant as expressed in

*The facilities of the Numerical Analysis Laboratory of the University of Wisconsin were utilized for this purpose.
equation (51) was used as a basis for numerical solution. However, due to inherent difficulties involving the subtraction of large numbers of almost equal magnitude, it was found that sufficient accuracy could not be obtained from this approach. The sixth order determinant was then reduced in literal form to the fourth order determinant shown on the following page. A procedure was devised from which the desired eigenvalue can be determined from this determinant by trial and error. For a given set of parameters containing the dimensions and physical constants of the cylinder, the assumed value of the eigenvalue is varied until the value of the determinant becomes equal to zero. In this procedure each of the twenty-four terms in the expansion of the fourth order determinant is computed independently and their sum is accumulated subsequently. This eliminates the previous difficulties in connection with loss of accuracy.

The numerical results obtained are shown in Tables 1 through 7. The values are believed to be correct to three significant figures. More significant figures can be obtained by simply increasing the number of trials used. Curves based on the values given in the tables are shown in figures 6, 7, and 8.
Discussion of Results

The curves shown in figures 6, 7, and 8 are sufficient to illustrate how the value of the critical load varies when the parameters representing the dimensions and physical constants of the sandwich cylinder are varied. The ratios, \( \frac{E_c}{G_{r\theta}} = 4 \) and \( \frac{E_c}{G_{rz}} = 10 \), were chosen since they represent average values of these quantities for honeycomb cores, the type most commonly used in sandwich construction. Computations were also made with the values of these ratios interchanged since either situation is possible depending upon the orientation of the core in fabrication. The numerical results shown in Tables 1 through 4 and the curves shown in figures 6 and 7 are based on the value of \( \beta = 1 \). This value represents a cylinder that has a weak core; that is, a core having relatively low values of \( E_c, G_{r\theta}, \) and \( G_{rz} \). The numerical results shown in Tables 5 through 7 and the curves shown in figure 8 are based on the value of \( \beta = 1,000 \). This value of \( \beta \) represents a cylinder having a stiff core. Since \( \beta \) depends not only upon the ratio, \( \frac{E_c}{E} \), but also upon the ratio, \( \frac{a}{b} \), the practical range of values of \( \beta \) is quite wide. Most constructions have dimensions and physical properties such that they lie in the range between \( \beta = 1 \) and \( \beta = 1,000 \). All of the computations are based on the value of .3 for the Poisson's ratio of the facings. Results were obtained for only two different ratios of the radius of the inner facing to the radius of the outer facing; that is, \( \frac{b}{a} = .95 \) and \( \frac{b}{a} = .98 \). The procedure for machine computation was set up so that any or all of the values of these parameters may be changed without any difficulty.
The results indicate that, for cylinders having relatively weak cores, it is advantageous to have the higher value of the modulus of rigidity of the core in the tangential direction rather than in the longitudinal direction. The effect of the interchange of these values is evident from a comparison of the curves shown in figures 6 and 7. This effect is negligible in the case of cylinders having strong cores as evidenced by the fact that the values given in Tables 5 and 6 are essentially the same. The fact that in all cases the overall stiffness properties of the core become relatively more important as the length of the cylinder decreases may also be noted. As the length of the cylinders is increased, the values obtained for the critical loads approach the values previously obtained on the assumption that the cylinder length is infinite. This is to be expected because, if \( \lambda \) is allowed to approach zero, the characteristic determinant for this problem reduces to the determinant obtained on the basis of the assumption of infinite length.* As the cylinder decreases in length, the value of the critical load approaches the value \( G_{r\theta} \left(1 - \frac{b}{a}\right) \) and then falls off rapidly to zero. In any particular case, if \( E_c \) is set equal to infinity, the critical load becomes equal to \( G_{r\theta} \left(1 - \frac{b}{a}\right) \) when the length becomes zero. This limiting value for the critical load is characteristic of stability analyses of sandwich construction in general, if \( E_c \) is assumed to be infinite. Examination of the curves shown in figures 6, 7, and 8 shows that the curves representing the cylinder having a weak core reach a maximum in the neighborhood

*See Ref. 2.
of \( \frac{l}{a} = 1 \) while those representing the cylinder having a strong core continue to rise until the cylinder becomes very short. Indications are that analyses based on the assumption of \( E_c = \infty \) are sufficiently accurate in most ranges, but for relatively short cylinders having weak cores such an assumption may result in serious error.

Conclusions

The stability analysis of sandwich cylinders subjected to uniform external lateral pressure presented here is believed to be sufficiently accurate for use in design. The assumption of membrane facings limits the range of applicability of the results somewhat since the effect of the stiffnesses of the individual facings may be appreciable in some sandwich cylinders, particularly short cylinders having relatively thick facings. However, for cylinders of usual sandwich construction and with a length equal to or greater than the radius, this theory is believed to be adequate. The analysis of the problem with the stiffnesses of the facings taken into account may be performed without a great amount of additional work, and such an analysis is planned for the near future. Additional curves giving the values of the critical loads for a greater number of values of the parameters entering the problem are desirable for design purposes. Such curves may be easily prepared in the future if the time and expense involved in their preparation seem justifiable. The Forest Products Laboratory is at present planning a series of tests for the purpose of correlating the results with the theoretical results obtained here.
Bibliography


Table 1. Values of Critical Pressure Expressed in Terms of $\alpha$.

$$\sigma = \frac{q_c r a (1 + \mu^2)}{Et} \quad \beta = 1$$

$$\lambda = \frac{\pi a}{l} \quad \frac{E_c}{Gr\theta} = 4$$

$$\frac{b}{a} = .95 \quad \frac{E_c}{Gr} = 10$$

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Table 2. Values of Critical Pressure Expressed in Terms of $\alpha$.

\[
\alpha = \frac{a_{cr} \alpha (1-\mu^2)}{Et}
\]

\[
\beta = 1
\]

\[
\lambda = \frac{\pi a}{l}
\]

\[
\frac{E_C}{G_{rz}} = 4
\]

\[
\frac{E_C}{G_{rz}} = 10
\]

\[
\frac{b}{a} = .98
\]

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Table 3. Values of Critical Pressure Expressed in Terms of $d$.

\[ d = \frac{q_{cr} a (1-\mu^2)}{Et} \]

\[ \lambda = \frac{\pi a}{l} \]

\[ \frac{Ec}{Gr} = 10 \]

\[ \frac{Ec}{Gr} = 4 \]

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Table 4. Values of Critical Pressure Expressed in Terms of $d$.

\[
\begin{align*}
  d &= \frac{d_{cr} a (1-\mu^2)}{Et} \\
  \lambda &= \frac{\pi a}{l} \\
  \frac{b}{a} &= .98 \\
  \beta &= 1 \\
  \frac{E_c}{G_{r\theta}} &= 10 \\
  \frac{E_c}{G_{rz}} &= 4
\end{align*}
\]

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Table 5. Values of Critical Pressure Expressed in Terms of $a$.

\[
\alpha = \frac{q_{cr} a (1-\mu^2)}{Et}
\]

\[
\beta = 1000
\]

\[
\lambda = \frac{\pi a}{l}
\]

\[
\frac{E_c}{G_r \theta} = 4
\]

\[
\frac{E_c}{G_r z} = 10
\]

\[
b = 0.95
\]

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Table 6. Values of Critical Pressure Expressed in Terms of \(a\).

\[ d = \frac{\alpha_{cr} a (1-\mu^2)}{Et} \]  
\[ \beta = 1000 \]
\[ \lambda' = \frac{\pi a}{l} \]
\[ \frac{E_c}{G_{yx}} = 4 \]
\[ \frac{E_c}{G_{xz}} = 10 \]

\[
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.25 & : & " & : & 0.000771 & : & 5.00 & : & 7 & : & 0.0265 \\
.40 & : & " & : & 0.00153 & : & 6.00 & : & " & : & 0.0361 \\
.50 & : & " & : & 0.00272 & : & 6.00 & : & 8 & : & 0.0349 \\
.50 & : & 3 & : & 0.00189 & : & 7.00 & : & " & : & 0.0452 \\
.60 & : & " & : & 0.00209 & : & 7.00 & : & 9 & : & 0.0448 \\
.75 & : & " & : & 0.0061 & : & 8.00 & : & " & : & 0.0562 \\
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3.15 & : & 6 & : & 0.0140 & : & " & : & " & : & \\
\end{array}
\]
Table 7. Values of Critical Pressure Expressed in Terms of $\alpha$.

$$d = \frac{q_{cr} a (1-\mu^2)}{Et}$$

$$\lambda = \frac{\pi a}{\ell}$$

$$\beta = 1000$$

$$\frac{E_c}{Gr_g} = 10$$

$$\frac{E_c}{Gr_z} = 4$$

$$b = 0.95$$

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Figure 1.—Sandwich cylinder.
Figure 2.—Differential elements of core and facings before buckling.
Figure 3.--Differential element of deformed core.
Figure 4.—Differential elements of deformed facings.
Figure 5.--Cross-section of buckled cylinder.

(for n = 4)
Figure 6.--Critical pressure in terms of $a_{cr}$ versus the length to radius ratio.
Figure 7.—Critical pressure in terms of $\alpha r$ versus the length to radius ratio.
Figure 8.—Critical pressure in terms of $a/c_r$ versus the length to radius ratio.