THEORETICAL DESIGN OF A NAILED OR BOLTED JOINT UNDER LATERAL LOAD

Revised March 1955

INFORMATION REVIEWED AND REAFFIRMED 1960

No. D1951

UNITED STATES DEPARTMENT OF AGRICULTURE
FOREST SERVICE
FOREST PRODUCTS LABORATORY
Madison 5, Wisconsin
In Cooperation with the University of Wisconsin
THEORETICAL DESIGN OF A NAILED OR BOLTED JOINT
UNDER LATERAL LOAD\(^1\)

By

EDWARD W. KUENZI,\(^2\) Engineer

Forest Products Laboratory,\(^3\) Forest Service
U. S. Department of Agriculture

Summary

This report presents a theoretical method for determining the allowable lateral load of a joint having a single nail or bolt. The design value of the joint is computed after combining various parameters involving the material properties of the members being joined and the joining nail or bolt. Included is an appendix giving several examples illustrating application of the theoretical method to specific joints.

Introduction

A great deal of experimental work has been done to evaluate the lateral resistance of nailed or bolted joints. Usually the final design value is arrived at for a particular joint by applying a factor of safety to the maximum test load or by determining the test load at some specified amount of joint slip or deformation. The former method may lead to structures in which joint deformations may be excessive. The latter method may result in joints in which local stresses in the members or in the nail or bolt will

\(^1\)Originally issued in September 1953.

\(^2\)The author wishes to acknowledge the assistance of W. S. Ericksen in formulating the problem and carrying out some of the mathematical calculations.

\(^3\)Maintained at Madison, Wis., in cooperation with the University of Wis.

Report No. 1951 Agriculture-Madison
exceed allowable values because the experimental determination of joint slip may have included much deformation in test members and apparatus; thus small local yielding of the joint would not appreciably affect the total deformation as measured, and a higher load, high enough so that local yielding would be great enough to cause nonlinearity, would be the apparent design value.

Certain empirical design criteria, in addition to simplifying assumptions, are in current use for the lateral resistance of nailed or bolted joints. It is felt, however, that an analysis including the properties of various materials would enable better design of joints. Such an analysis could also be applied to materials in which no actual joint test data are available for the particular sizes and materials that it is desired to use.

In 1951 Möller described the failure of a nailed joint by considering the nail under uniformly distributed bearing forces equal to the strength of the wood. His analysis, however, did not consider joint deformation or the possibility of the nail being permanently distorted at stresses in the wood less than the strength of the wood.

The following analysis considers the nail or bolt as being supported on an elastic foundation. The design value for the joint is finally found by determining the load at which stresses in either the members or the bolts reach proportional limit or yield values, whichever produces the least load. Expressions are given for determining the deformation of the joint at any load up to this yielding joint load. Extrapolation of the analysis to greater loads is not expected to produce correct results unless some method be devised to take into consideration the plastic behavior of the members and the nails or bolts.

Theoretical Design Criteria

The nail or bolt is assumed to be a beam supported on an elastic foundation such that deflection of the nail or bolt is resisted by a pressure proportional to the deflection at any point and such that this pressure can be exerted in both upward and downward directions. The analysis of beams on elastic foundations of this type was first introduced by E. Winkler in 1867, and solutions for beams of finite length that are applied here were given by Hetényi in 1946. The foundation is assumed to be discontinuous (transmitting no shear), but it has been found that moments produced in


beams supported on an elastic solid may be closely approximated by considering the elastic solid to be an elastic foundation with a foundation modulus equal to its modulus of elasticity \( \frac{\text{beam width}}{\text{foundation depth}} \).

Additional assumptions in regard to the joint itself are that the load due to friction between the members is negligible and that tractions are not developed between the members and the nail or bolt. Both of these assumptions are somewhat incorrect when joints between wood members are first made, but in use the changes in moisture in the wood will cause slight loosening of the joint and thus rectify the assumptions. It is common practice in testing joints to allow for shrinkage by spacing the members slightly and be not joining tightly.

The fit between the bolt and hole is considered to be tight. The hole should be smooth in order that the joint will function most efficiently. It has been demonstrated that a rough hole will lower the proportional limit value for a joint in wood to about one-third of the value obtained for a bolt in a smooth hole.

The differential equation for the deflection curve of a beam supported on an elastic foundation is given by

\[
EI \frac{d^4 y}{dx^4} = - ky
\]

where \( EI \) = stiffness of the beam; \( E \) = modulus of elasticity; \( I \) = moment of inertia

\( y \) = deflection at point \( x \)

\( k \) = foundation modulus.

The solution of equation 1 finally results in expressions involving a characteristic

\[
\lambda = \sqrt{\frac{k}{4EI}}
\]

and expressions for deflections, moments, and shears depend upon the value of this characteristic.

---

Solutions for two types of joints follow.

I. Nailed or Bolted Joint of Two Members

The joint considered is shown in Figure 1. Figure 1, A, shows the choice of axes and Figure 1, B, shows the forces at the joint and the slip between the joints. The thicknesses of the members are designated by \(a\) and \(b\). If the nail or bolt does not completely penetrate the member, \(a\) or \(b\) would be the depth of penetration.

The deflection of the nail or bolt in member 1 is given by

\[
y_1 = - \frac{2P\lambda_1}{\Delta_1 k_1} \left\{ \text{Sinh} \lambda_1 a \cos \lambda_1 (a - x) - \sin \lambda_1 a \cos \lambda_1 x \cosh \lambda_1 (a - x) \right\}
\]

\[
+ \frac{2M_0 \lambda_1^2}{\Delta_1 k_1} \left\{ \text{Sinh} \lambda_1 a \left[ \cosh \lambda_1 x \sin \lambda_1 (a - x) - \text{Sinh} \lambda_1 x \cos \lambda_1 (a - x) \right] \right\}
\]

\[
+ \sin \lambda_1 a \left[ \text{Sinh} \lambda_1 (a - x) \cos \lambda_1 x - \text{Cosh} \lambda_1 (a - x) \sin \lambda_1 x \right] \right\} \quad (3)
\]

where

\[
\lambda_1 = \sqrt{\frac{k_1}{4EI}}
\]

\[
\Delta_1 = \text{Sinh}^2 \lambda_1 a - \sin^2 \lambda_1 a
\]

The slope of the nail or bolt in member 1 is given by

\[
\frac{dy_1}{dx} = - \frac{2P\lambda_1^2}{\Delta_1 k_1} \left\{ \text{Sinh} \lambda_1 a \left[ \cosh \lambda_1 x \sin \lambda_1 (a - x) + \text{Sinh} \lambda_1 x \cos \lambda_1 (a - x) \right] \right\}
\]

\[
+ \sin \lambda_1 a \left[ \cos \lambda_1 x \text{Sinh} \lambda_1 (a - x) + \sin \lambda_1 x \text{Cosh} \lambda_1 (a - x) \right] \right\}
\]

\[
- \frac{4M_0 \lambda_1^3}{\Delta_1 k_1} \left\{ \text{Sinh} \lambda_1 a \cosh \lambda_1 x \cos \lambda_1 (a - x) + \sin \lambda_1 a \cos \lambda_1 x \cosh \lambda_1 (a - x) \right\} \quad (4)
\]
By substituting the expression for $\lambda_1$, and further recalling that the moment $M_1 = -EI \frac{d^2y_1}{dx^2}$, the moment at any point along the nail or bolt in member 1 is given by

$$M_1 = \frac{P}{\Delta_1 \lambda_1} \left\{ \sinh \lambda_1 a \sinh \lambda_1 x \sin \lambda_1(a-x) - \sin \lambda_1 a \sin \lambda_1 x \sinh \lambda_1(a-x) \right\} + \frac{M_0}{\Delta_1} \left\{ \sinh \lambda_1 a \left[ \sinh \lambda_1 x \cos \lambda_1(a-x) + \cosh \lambda_1 x \sin \lambda_1(a-x) \right] \right. \\
\left. - \sin \lambda_1 a \left[ \cos \lambda_1 x \sin \lambda_1(a-x) + \sin \lambda_1 x \cosh \lambda_1(a-x) \right] \right\} \quad (5)$$

By recalling that the shear $Q_1 = \frac{dM_1}{dx}$, the shear at any point along the nail or bolt in member 1 is given by

$$Q_1 = \frac{P}{\Delta_1} \left\{ \sinh \lambda_1 a \left[ \cosh \lambda_1 x \sin \lambda_1(a-x) - \sinh \lambda_1 x \cos \lambda_1(a-x) \right] \right. \\
\left. - \sin \lambda_1 a \left[ \cos \lambda_1 x \sin \lambda_1(a-x) - \sin \lambda_1 x \cosh \lambda_1(a-x) \right] \right\} + \frac{2M_0 \lambda_1}{\Delta_1} \left\{ \sinh \lambda_1 a \sin \lambda_1 x \sin \lambda_1(a-x) + \sin \lambda_1 a \sin \lambda_1 x \sinh \lambda_1(a-x) \right\} \quad (6)$$

Similar expressions will be written for the deflections, slopes, moments, and shears in the bolt in member 2.

$$y_2 = \frac{2P \lambda_2^2}{\Delta_2^2 k_2} \left\{ \sinh \lambda_2 b \cos \lambda_2 x \cosh \lambda_2(b-x) - \sin \lambda_2 b \cosh \lambda_2 x \cos \lambda_2(b-x) \right\} + \frac{2M_0 \lambda_2^2}{\Delta_2^2 k_2} \left\{ \sinh \lambda_2 b \left[ \sin \lambda_2 x \cosh \lambda_2(b-x) - \cos \lambda_2 x \sinh \lambda_2(b-x) \right] \right. \\
\left. + \sin \lambda_2 b \left[ \sinh \lambda_2 x \cos \lambda_2(b-x) - \cosh \lambda_2 x \sin \lambda_2(b-x) \right] \right\} \quad (7)$$
where

\[ \Delta_2 = \sinh^2 \lambda_2 b - \sin^2 \lambda_2 b \]

\[ \lambda_2 = \sqrt{\frac{k_2}{4EI}} \]

\[
\frac{dy_2}{dx} = -\frac{2P\lambda_2^2}{\Delta_2 k_2} \left\{ \text{Sinh} \lambda_2 b \left[ \sin \lambda_2 x \cosh \lambda_2 (b - x) + \cos \lambda_2 x \sinh \lambda_2 (b - x) \right] \right.

\[ + \sin \lambda_2 b \left[ \text{Sinh} \lambda_2 x \cos \lambda_2 (b - x) + \text{Cosh} \lambda_2 x \sin \lambda_2 (b - x) \right] \}

\[ + \frac{4M_0 \lambda_2^3}{\Delta_2 k_2} \left\{ \text{Sinh} \lambda_2 b \cos \lambda_2 x \cosh \lambda_2 (b - x) + \sin \lambda_2 b \text{Cosh} \lambda_2 x \cos \lambda_2 (b - x) \right\} \]

(8)

\[
M_2 = -\frac{P_2}{\Delta_2 \lambda_2} \left\{ \text{Sinh} \lambda_2 b \sin \lambda_2 x \text{Sinh} \lambda_2 (b - x) - \sin \lambda_2 b \sinh \lambda_2 x \sin \lambda_2 (b - x) \right\}

\[ + \frac{M_0}{\Delta_2} \left\{ \text{Sinh} \lambda_2 b \left[ \cos \lambda_2 x \text{Sinh} \lambda_2 (b - x) + \sin \lambda_2 x \text{Cosh} \lambda_2 (b - x) \right] \right. \]

\[ - \sin \lambda_2 b \left[ \text{Sinh} \lambda_2 x \cos \lambda_2 (b - x) + \text{Cosh} \lambda_2 x \sin \lambda_2 (b - x) \right] \}

(9)

\[
Q_2 = -\frac{P}{\Delta_2} \left\{ \text{Sinh} \lambda_2 b \left[ \cos \lambda_2 x \text{Sinh} \lambda_2 (b - x) - \sin \lambda_2 x \text{Cosh} \lambda_2 (b - x) \right] \right.

\[ - \sin \lambda_2 b \left[ \text{Cosh} \lambda_2 x \sin \lambda_2 (b - x) - \text{Sinh} \lambda_2 x \cos \lambda_2 (b - x) \right] \}

\[ - \frac{2M_0 \lambda_2}{\Delta_2} \left\{ \text{Sinh} \lambda_2 b \sin \lambda_2 x \text{Sinh} \lambda_2 (b - x) + \sin \lambda_2 b \text{Sinh} \lambda_2 x \sin \lambda_2 (b - x) \right\} \]

(10)
At the juncture of members 1 and 2, the slope of the nail or bolt are the same and

\[
\frac{dy_1}{dx} \bigg|_{x = a} = \frac{dy_2}{dx} \bigg|_{x = 0}
\]  

(11)

Substitution of equations 4 and 8 in equation 11 gives

\[
-\frac{2\lambda_1^2}{\Delta_1 k_1} \left[ \text{Sinh}^2 \lambda_1 a + \sin \lambda_1 a \right] - \frac{4M_0 \lambda_1^3}{\Delta_1 k_1} \left[ \text{Sinh} \lambda_1 a \text{Cosh} \lambda_1 a + \sin \lambda_1 a \cos \lambda_1 a \right] =
\]

\[
-\frac{2\lambda_2^2}{\Delta_2 k_2} \left[ \text{Sinh}^2 \lambda_2 b + \sin \lambda_2 b \right] + \frac{4M_0 \lambda_2^3}{\Delta_2 k_2} \left[ \text{Sinh} \lambda_2 b \text{Cosh} \lambda_2 b + \sin \lambda_2 b \cos \lambda_2 b \right]
\]

(12)

Let

\[
J_1 = \frac{\lambda_1^2}{k_1 \Delta_1 k_1} \left[ \text{Sinh}^2 \lambda_1 a + \sin \lambda_1 a \right] \left[ \text{Sinh} \lambda_1 a - \sin \lambda_1 a \right]
\]

(13)

\[
J_2 = \frac{\lambda_2^2}{k_2 \Delta_2 k_2} \left[ \text{Sinh}^2 \lambda_2 b + \sin \lambda_2 b \right] \left[ \text{Sinh} \lambda_2 b - \sin \lambda_2 b \right]
\]

\[
K_1 = \frac{\lambda_1^3}{k_1 \Delta_1 k_1} \left[ \text{Sinh} \lambda_1 a \text{Cosh} \lambda_1 a + \sin \lambda_1 a \cos \lambda_1 a \right] \left[ \text{Sinh} \lambda_1 a - \sin \lambda_1 a \right]
\]

\[
K_2 = \frac{\lambda_2^3}{k_2 \Delta_2 k_2} \left[ \text{Sinh} \lambda_2 b \text{Cosh} \lambda_2 b + \sin \lambda_2 b \cos \lambda_2 b \right] \left[ \text{Sinh} \lambda_2 b - \sin \lambda_2 b \right]
\]

(Curves for use in computing \( J \) and \( K \) are given in figure 2.)

Report No. 1951 -7-
Then equation 12 becomes

\[-2pJ_1 - 4M_0K_1 = -2pJ_2 + 4M_0K_2\]

and finally

\[M_0 = -\frac{p(J_1 - J_2)}{2(K_1 + K_2)}\]  \hspace{1cm} (14)

Also at the juncture of members 1 and 2 the slip, \(\delta\), is given by

\[\delta = -y_1 \bigg|_{x=a} + y_2 \bigg|_{x=0}\]  \hspace{1cm} (15)

and substitution of equations 3 and 7 in equation 15 gives

\[
\delta = \frac{2p\lambda_1}{\Delta_1 k_1} \left[ \sinh \lambda_1 a \cosh \lambda_1 a - \sin \lambda_1 a \cos \lambda_1 a \right] \\
+ \frac{2M_0 \lambda_1^2}{\Delta_1 k_1} \left[ \sinh^2 \lambda_1 a + \sin \lambda_1 a \cos \lambda_1 a \right] \\
+ \frac{2p\lambda_2}{\Delta_2 k_2} \left[ \sinh \lambda_2 b \cosh \lambda_2 b - \sin \lambda_2 b \cos \lambda_2 b \right] \\
- \frac{2M_0 \lambda_2^2}{\Delta_2 k_2} \left[ \sinh^2 \lambda_2 b + \sin \lambda_2 b \cos \lambda_2 b \right] \]  \hspace{1cm} (16)

Let

\[L_1 = \frac{\lambda_1}{k_1} \left[ \frac{\sinh \lambda_1 a \cosh \lambda_1 a - \sin \lambda_1 a \cos \lambda_1 a}{\sinh^2 \lambda_1 a - \sin^2 \lambda_1 a} \right] \]  \hspace{1cm} (17)

\[L_2 = \frac{\lambda_2}{k_2} \left[ \frac{\sinh \lambda_2 b \cosh \lambda_2 b - \sin \lambda_2 b \cos \lambda_2 b}{\sinh^2 \lambda_2 b - \sin^2 \lambda_2 b} \right] \]

(Curves for use in computing \(L\) are given in figure 2)

Report No. 1951
Substituting equations 13, 14, and 17 in 16 finally results in

\[ \delta = P \left[ \frac{2(L_1 + L_2) - (J_1 - J_2)^2}{K_1 + K_2} \right] \]  

(18)

The pressure under the nail or bolt at any point is given by

\[ p = ky \]

Therefore, in member 1 the maximum pressure will occur at \( x = a \) and will be equal to

\[ p_{1 \text{ max.}} = -P_k \left[ \frac{J_1(J_1 - J_2)}{2L_1 - (J_1 - J_2)} \right] \]  

(19)

and in member 2 the maximum pressure will occur at \( x = 0 \) and will be equal to

\[ p_{2 \text{ max.}} = P_k \left[ \frac{J_1(J_1 - J_2)}{2L_2 + (J_1 - J_2)} \right] \]  

(20)

The maximum moment in the nail or bolt occurs in either member 1 or 2 at a point where the shear is zero. By setting the equation for the shear in member 1 (equation 6) equal to zero, after expansion and final reduction the shear will be zero at values of \( x \) satisfying the equation

\[ \coth \lambda_1 x + \cot \lambda_1 x = \frac{2 - B_1[\coth \lambda_1 a + \cot \lambda_1 a]}{\coth \lambda_1 a - \cot \lambda_1 a - B_1} \]  

(21)

where

\[ B_1 = \frac{\lambda_1(J_1 - J_2)}{K_1 + K_2} \]

Similarly, the shear in the nail or bolt in member 2 will vanish at values of \( x \) satisfying the equation
\[ \text{Coth } \lambda_2(b-x) + \cot \lambda_2(b-x) = \frac{2 + B_2 \left[ \text{Coth } \lambda_2 b + \cot \lambda_2 b \right]}{\text{Coth } \lambda_2 b - \cot \lambda_2 b + B_2} \]  

(22)

where

\[ B_2 = \frac{\lambda_2(J_1 - J_2)}{K_1 + K_2} \]

A curve of the value of Coth \( \theta \) + cot \( \theta \) is given in figure 3 for use in solving for \( \theta = \lambda_1 x \) or \( \theta = \lambda_2(b-x) \) in equations 21 and 22, respectively. The function Coth \( \theta \) + cot \( \theta \) is periodic, and only one branch to \( \theta = \pi \) is shown in figure 3. For solutions beyond \( \theta = \pi \) the Coth \( \theta \) may be assumed to be one without great error and then equations 21 and 22 can be solved directly for cot \( \lambda_1 x \) or cot \( \lambda_2(b-x) \).

II. Nailed or Bolted Joint of Three Members

The joint considered is shown in figure 4. Figure 4, A, shows the choice of axes, and figure 4, B, shows the forces at the joint and the slip between joints. The thickness of the inner member is designated by \( a \); each of the outer members, of equal thickness, are of thickness \( b \).

The deflection of the nail or bolt in member 1 is given by

\[ y_1 = -\frac{P\lambda_1}{\Delta_1 k_1} \left\{ \text{Cosh } \lambda_1 x \cos \lambda_1(a-x) + \cos \lambda_1 x \text{Cosh } \lambda_1(a-x) \right\} - \frac{2M_0\lambda_1^2}{\Delta_1 k_1} \left\{ \sinh \lambda_1 x \cos \lambda_1(a-x) - \text{Cosh } \lambda_1 x \sin \lambda_1(a-x) + \cos \lambda_1 x \text{Sinh } \lambda_1(a-x) - \sin \lambda_1 x \text{Cosh } \lambda_1(a-x) \right\} \]

(23)

Where

\[ \lambda_1 = \sqrt{\frac{k_1}{4E_1}} \]

\[ \Delta_1 = \text{Sinh } \lambda_1 a + \sin \lambda_1 a \]
The slope of the nail or bolt in member 1 is given by

\[
\frac{dy_1}{dx} = -\frac{P\lambda_1^2}{\Delta_1 k_1} \left\{ \text{Sinh } \lambda_1 x \cos \lambda_1 (a - x) + \text{Cosh } \lambda_1 x \sin \lambda_1 (a - x) \right. \\
\left. - \cos \lambda_1 x \text{Sinh } \lambda_1 (a - x) - \sin \lambda_1 x \text{Cosh } \lambda_1 (a - x) \right\} \\
- \frac{4M_0 \lambda_1^3}{\Delta_1 k_1} \left\{ \text{Cosh } \lambda_1 x \cos \lambda_1 (a - x) - \cos \lambda_1 x \text{Cosh } \lambda_1 (a - x) \right\} 
\] (24)

The moment at any point along the nail or bolt in member 1 is given by

\[
M_1 = \frac{P}{2\Delta_1 \lambda_1} \left\{ \text{Sinh } \lambda_1 x \sin \lambda_1 (a - x) + \sin \lambda_1 x \text{Sinh } \lambda_1 (a - x) \right\} \\
+ \frac{M_0}{\Delta_1} \left\{ \text{Sinh } \lambda_1 x \cos \lambda_1 (a - x) + \text{Cosh } \lambda_1 x \sin \lambda_1 (a - x) \right\} \\
+ \cos \lambda_1 x \text{Sinh } \lambda_1 (a - x) + \sin \lambda_1 x \text{Cosh } \lambda_1 (a - x) 
\] (25)

The shear at any point along the nail or bolt in member 1 is given by

\[
Q_1 = -\frac{P}{2\Delta_1} \left\{ \text{Sinh } \lambda_1 x \cos \lambda_1 (a - x) - \text{Cosh } \lambda_1 x \sin \lambda_1 (a - x) \right\} \\
+ \sin \lambda_1 x \text{Cosh } \lambda_1 (a - x) - \cos \lambda_1 x \text{Sinh } \lambda_1 (a - x) \right\} \\
+ \frac{2M_0 \lambda_1}{\Delta_1} \left\{ \text{Sinh } \lambda_1 x \sin \lambda_1 (a - x) - \sin \lambda_1 x \text{Sinh } \lambda_1 (a - x) \right\} 
\] (26)

The expressions for the deflection, slope, moment, and shear in member 2 will be identical with equations 7, 8, 9, and 10, but with \( P \) in those equations replaced by \( P/2 \).

As before at the juncture of members 1 and 2, the slope of the nail or bolt are the same and

\[
\frac{dy_1}{dx} \bigg|_{x = a} = \frac{dy_2}{dx} \bigg|_{x = 0} 
\] (27)
Substitution of equations 24 and modified equation 8 into equation 27 gives

\[
- \frac{P \lambda_1^2}{\Delta_1 k_1} \left[ \sinh \lambda_1 a - \sin \lambda_1 a \right] - \frac{4M_0 \lambda_1^3}{\Delta_1 k_1} \left[ \cosh \lambda_1 a - \cos \lambda_1 a \right] =
\]

\[
- \frac{P \lambda_2^2}{\Delta_2 k_2} \left[ \sinh^2 \lambda_2 b + \sin^2 \lambda_2 b \right] + \frac{4M_0 \lambda_2^3}{\Delta_2 k_2} \left[ \sinh \lambda_2 b \cosh \lambda_2 b + \sin \lambda_2 b \cos \lambda_2 b \right]
\]

(28)

Let

\[
J_1 = \frac{\lambda_1^2}{k_1} \left[ \frac{\sinh \lambda_1 a - \sin \lambda_1 a}{\sinh \lambda_1 a + \sin \lambda_1 a} \right]
\]

\[
J_2 = \frac{\lambda_2^2}{k_2} \left[ \frac{\sinh^2 \lambda_2 b + \sin^2 \lambda_2 b}{\sinh^2 \lambda_2 b - \sin^2 \lambda_2 b} \right]
\]

\[
K_1 = \frac{\lambda_1^3}{k_1} \left[ \frac{\cosh \lambda_1 a - \cos \lambda_1 a}{\sinh \lambda_1 a + \sin \lambda_1 a} \right]
\]

\[
K_2 = \frac{\lambda_2^3}{k_2} \left[ \frac{\sinh \lambda_2 b \cosh \lambda_2 b + \sin \lambda_2 b \cos \lambda_2 b}{\sinh^2 \lambda_2 b - \sin^2 \lambda_2 b} \right]
\]

(29)

(Curves for use in computing \(J_1\) and \(K_1\) are given in figure 5, for \(J_2\) and \(K_2\) in figure 2.)

Then equation 28 becomes

\[- PJ_1 - 4M_0 K_1 = - PJ_2 + 4M_0 K_2\]
and, finally,

\[ M_0 = -\frac{P(J_1 - J_2)}{4(K_1 + K_2)} \]  

(30)

Also at the juncture of members 1 and 2, the slip, \( \delta \), is given by

\[ \delta = -\frac{y_1}{x=a} + \frac{y_2}{x=0} \]  

(31)

and substitution of equations 23 and modified equation 7 in equation 31 gives

\[
\begin{align*}
\delta &= \frac{P\lambda_1}{\Delta_1 k_1} \left[ \cosh \lambda_1 a + \cos \lambda_1 a \right] + \frac{2M_0 \lambda_1^2}{\Delta_1 k_1} \left[ \sinh \lambda_1 a - \sin \lambda_1 a \right] \\
&\quad + \frac{P\lambda_2}{\Delta_2 k_2} \left[ \sinh \lambda_2 b \cosh \lambda_2 b - \sin \lambda_2 b \cos \lambda_2 b \right] \\
&\quad - \frac{2M_0 \lambda_2^2}{\Delta_2 k_2} \left[ \sinh^2 \lambda_2 b + \sin^2 \lambda_2 b \right]
\end{align*}
\]  

(32)

Let

\[
\begin{align*}
L_1 &= \frac{\lambda_1}{k_1} \left[ \frac{\cosh \lambda_1 a + \cos \lambda_1 a}{\sinh \lambda_1 a + \sin \lambda_1 a} \right] \\
L_2 &= \frac{\lambda_2}{k_2} \left[ \frac{\sinh \lambda_2 b \cosh \lambda_2 b - \sin \lambda_2 b \cos \lambda_2 b}{\sinh^2 \lambda_2 b - \sin^2 \lambda_2 b} \right]
\end{align*}
\]  

(33)

(Curves for use in computing \( L_1 \) are given in figure 5, for \( L_2 \) in figure 2.)
Substituting equations 29, 30, and 33 in 32 finally results in

\[ \delta = P \left[ L_1 + L_2 - \frac{(J_1 - J_2)^2}{2(K_1 + K_2)} \right] \]  \hspace{1cm} (34)

The pressure under the nail or bolt at any point is given by

\[ p = ky \]

Therefore, in member 1, the maximum pressure will occur at \( x = a \) or \( x = 0 \) and will be equal to

\[ p_{1 \text{ max.}} = - Pk_1 \left[ L_1 - \frac{J_1(J_1 - J_2)}{2(K_1 + K_2)} \right] \]  \hspace{1cm} (35)

and in member 2 the maximum pressure will occur at \( x = 0 \) and will be equal to

\[ p_{2 \text{ max.}} = Pk_2 \left[ L_2 + \frac{J_2(J_1 - J_2)}{2(K_1 + K_2)} \right] \]  \hspace{1cm} (36)

As for the joint of two members, the maximum moment in the nail or bolt occurs in either member 1 or 2 at the point where the shear is zero. By setting the expression for the shear in member 1 (equation 26) equal to zero, after expansion and final reduction the shear will be zero at \( x = a/2 \) or at values of \( x \) satisfying the equation

\[ \frac{\cot \lambda_1(x - \frac{a}{2})}{\coth \lambda_1(x - \frac{a}{2})} = \frac{1 + \coth \lambda_1 \frac{a}{2} \cot \lambda_1 \frac{a}{2} + B_1 \cot \lambda_1 \frac{a}{2}}{1 - \coth \lambda_1 \frac{a}{2} \cot \lambda_1 \frac{a}{2} + B_1 \coth \lambda_1 \frac{a}{2}} \]  \hspace{1cm} (37)

where

\[ B_1 = \frac{\lambda_1(J_1 - J_2)}{K_1 + K_2} \]
A curve of \( \frac{\cot \theta}{\coth \theta} \) is given in figure 6 for use in solving for \( \theta = \lambda_1 (x - \frac{a}{2}) \) in equation 37. The function \( \frac{\cot \theta}{\coth \theta} \) is periodic, and only one branch to \( \theta = \pi \) is shown in figure 6. For solutions beyond \( \theta = \pi \) the \( \coth \theta \) may be assumed to be one without great error and then equation 37 can be solved directly for \( \cot \lambda_1 (x - \frac{a}{2}) \).

The position of zero shear is given in member 2 by the value of \( x \) satisfying the equation

\[
\coth \lambda_2 (b - x) + \cot \lambda_2 (b - x) = \frac{2 + B_2 \left[ \coth \lambda_2 b + \cot \lambda_2 b \right]}{\coth \lambda_2 b - \cot \lambda_2 b + B_2}
\]

where

\[
B_2 = \frac{\lambda_2 (J_1 - J_2)}{K_1 + K_2}
\]

A curve of the value of \( \coth \theta + \cot \theta \) is given in figure 3 for use in solving equation 38 for \( \theta = \lambda_2 (b - x) \). It should be recalled, as for the two-member joint, the \( \coth \theta + \cot \theta \) is a periodic function.

**Application of Theoretical Design Criteria**

For determining the design value of a joint, it is necessary to examine all possible points where high stress can begin to cause failure. Thus, a complete analysis must be carried out for any particular joint. Perhaps after calculating a series of joints involving certain changes of members or nails or bolts it may be possible to present curves of design values dependent on the various parameters involved.

For members having definite compressive failing stresses, such as wood loaded in a direction parallel to the grain, the load at which the compressive proportional limit stress is reached at the point of maximum nail or bolt deflection will often be the design value of the joint if the nail or bolt is relatively stiff. The foundation modulus, \( k \), for wood loaded parallel to the grain direction may be taken as the compressive modulus of elasticity \( x \frac{\text{beam width}}{\text{foundation depth}} \) because deformations occur mainly in directions parallel to the grain of the wood. The load at which yielding
of the nail or bolt occurs, due to maximum moment, should be calculated to be sure it is not less than the load necessary to exceed the compressive proportional limit stress of the member.

For members having no definite compressive failing stress, such as wood loaded in a direction perpendicular to the grain, the design value of the joint will be the load necessary to cause yielding of the nail or bolt. It should be remembered that some permanent joint slip due to plastic deformation of the members may occur at this design load. It might be expected that the nail or bolt loaded in a direction perpendicular to the grain would be supported by a foundation modulus in excess of the compressive modulus of elasticity perpendicular to the grain because of some supporting action in a direction parallel to the grain. Previous tests, however, have shown such supporting action to be insignificant, so that the foundation modulus may be taken to be the compressive modulus of elasticity perpendicular to the grain x beam width / foundation depth.

---

Appendix

Example 1

Determine the design value of a two-member joint of nominal 1- and 2-inch Douglas-fir with an eightpenny common nail. The joint is loaded laterally and in directions parallel to the grain in each member.

Members -- \( a = 1.625 \) inch, \( b = 0.781 \) inch, \( E = 2,112,000 \) pounds per square inch, \( S_{PL} = 6,450 \) pounds per square inch.

Nail -- \( d = 0.131 \) inch, \( E = 30,000,000 \) pounds per square inch, \( EI = 434 \) lb.-in.\(^2\), \( S_{EL} = 60,000 \) pounds per square inch, \( M_{EL} = 13.25 \) in.-lb.

\[
k = \frac{0.131}{1} \times 2,112,000 = 277,000 \text{ pounds per square inch}
\]
(assuming effective foundation depth of 1 inch).

\[
\lambda_1 = \lambda_2 = 3.55 \quad \text{then} \quad \lambda_1 a = 5.77 \quad \text{and} \quad \lambda_2 b = 2.77
\]

\[
\sinh \lambda_1 a = 160.3, \quad \cosh \lambda_1 a = 160.3, \quad \coth \lambda_1 a = 1.000, \quad \sin \lambda_1 a = -0.490
\]

\[
\cos \lambda_1 a = 0.873, \quad \cot \lambda_1 a = -1.783
\]

\[
\sinh \lambda_2 b = 7.948, \quad \cosh \lambda_2 b = 8.011, \quad \coth \lambda_2 b = 1.008, \quad \sin \lambda_2 b = 0.361
\]

\[
\cos \lambda_2 b = -0.932, \quad \cot \lambda_2 b = -2.583
\]

Then

\[
J_1 = \frac{\lambda_1^2}{k_1}, \quad J_2 = \frac{\lambda_2^2}{k_2}, \quad K_1 = \frac{\lambda_1^3}{k_1}, \quad K_2 = \frac{\lambda_2^3}{k_2}
\]

\[
L_1 = \frac{\lambda_1}{k_1}, \quad L_2 = \frac{\lambda_2}{k_2}
\]
and since

\[ \lambda_1 = \lambda_2 \quad \text{and} \quad k_1 = k_2 \]

then

\[ J_1 = J_2 \quad \text{and} \quad M_0 = 0 \quad \text{and} \quad B_1 = B_2 = 0 \]

Finally,

\[ P_{1\text{ max}} = -P_{2\text{ max}} = -2P\lambda \]

then

\[ P = \frac{P_{\text{max}}}{2 \times 3.55} = \frac{6450 \times 1.31}{7.10} = 119 \text{ pounds.} \]

In member 1 the shear is zero if

\[ \text{Coth } \lambda_1 x + \cot \lambda_1 x = \frac{2}{1 + 1.783} = 0.719 \]

and from figure 3, \( \theta = 1.88 \), \( \lambda_1 = 3.55 \), then \( x = 0.530 \), which is obviously not the position of maximum moment since the maximum moment will occur nearer to the joint. Since \( \lambda_1 x \) must be greater than 4, then \( \text{Coth } \lambda_1 x = 1 \), and finally

\[ \cot \lambda_1 x = -0.279 \]

which has solutions of

\[ \lambda_1 x = 4.986 \quad \text{and} \quad x = 1.405 \]

\[ \lambda_1 x = 8.127 \quad \text{and} \quad x = 2.29, \text{ which is physically impossible; therefore the maximum moment occurs at } x = 1.405. \]

\[ \lambda_1 x = 4.986, \quad \text{then } \text{Sinh } \lambda_1 x = 73.17 \quad \text{sin } \lambda_1 x = -0.963 \]

\[ \lambda_1 (a - x) = 0.780, \quad \text{then } \text{Sinh } \lambda_1 (a - x) = 0.861 \quad \text{sin } \lambda_1 (a - x) = 0.703 \]

and finally

\[ M_{1\text{ max}} = \frac{73.16 \times 0.703}{160.2 \times 3.55} = \frac{P}{11.1} \]

Report No. 1951 -18-
and

\[ P = 11.1 \times 13.25 = 147 \text{ pounds} \]

In member 2 the shear is zero if

\[ \text{Coth } \lambda_2 (b - x) + \cot \lambda_2 (b - x) = \frac{2}{1.008 + 2.583} = 0.557 \]

and from figure 3

\[ \theta = 2.02, \text{ then } \lambda_2 = 3.55, \ b - x = 0.569, \ x = 0.212, \]

Another solution is obtained for \( \theta = 5.127 \) or \( x = b - 1.445 \), which is physically impossible.

Then the maximum moment occurs at \( x = 0.212 \) and

\[ \lambda_2 x = 0.754 \quad \text{Sinh } \lambda_2 x = 0.828 \quad \sin \lambda_2 x = 0.685 \]

\[ \lambda_2 (b - x) = 2.02 \quad \text{Sinh } \lambda_2 (b - x) = 3.703 \quad \sin \lambda_2 (b - x) = 0.900 \]

and finally

\[ M_2 \text{ max.} = \frac{-3.703 \times 0.685}{7.948 \times 3.55} = \frac{-P}{11.1} \]

and

\[ P = 11.1 \times 13.25 = 147 \text{ pounds.} \]

Therefore the design value for the joint is determined by the compression produced in the wood and the design load is

\[ P = 119 \text{ pounds.} \]

At this load a joint slip of

\[ \delta = \frac{119 \times 4 \times 3.55}{277,000} = 0.0061 \text{ inch} \]

will occur.

---

8 Experiments often give several times this slip, but much of the deformation of the members and apparatus is usually included in such test data.

Report No. 1951 -19-
Figure 7 shows the shear, moment, and deflection diagrams for this example.

**Example 2**

Determine the design value of a three-member joint of nominal 2-inch Douglas-fir members on each side of a nominal 4-inch Douglas-fir member. The joint is made with a 1/2-inch steel machine bolt. The 2-inch side members are loaded parallel to the grain direction and the center member is loaded perpendicular to the grain direction.

Member 1 -- \( a = 3.625 \) inch, \( E_1 = 105,000 \) pounds per square inch

Member 2 -- \( b = 1.625 \) inch, \( E_2 = 2,112,000 \) pounds per square inch, \( S_{PL} = 6,450 \) pounds per square inch

Bolt -- \( E = 30,000,000 \) pounds per square inch, \( E_1 = 92,100 \) lb. -in. \(^2\),

\( S_{EL} = 30,000 \) pounds per square inch, \( M_{EL} = 368 \) in.-lb.

Then

\[
\lambda_1 = \sqrt{\frac{52,500}{4 \times 92,100}} = 0.615 \quad \lambda_1 a = 2.23
\]

\[
\sinh \lambda_1 a = 4.596 \quad \cosh \lambda_1 a = 4.704 \quad \coth \frac{\lambda_1 a}{2} = 1.240
\]

\[
\sin \lambda_1 a = 0.789 \quad \cos \lambda_1 a = -0.614 \quad \cot \frac{\lambda_1 a}{2} = 0.491
\]

\[
J_1 = \frac{0.378 \times 3.807}{52,500 \times 5.385} = 5.10 \times 10^{-6}
\]

\[
K_1 = \frac{0.232 \times 5.318}{52,500 \times 5.385} = 4.36 \times 10^{-6}
\]
\[ L_1 = \frac{0.615 \times 4.090}{52,500 \times 5.385} = 8.91 \times 10^{-6} \]

\[ k_2 = 2,112,000 \times \frac{0.50}{1} = 1,056,000 \text{ pounds per square inch} \]

\[ \lambda_2 = \sqrt{\frac{1,056,000}{4 \times 92,100}} = 1.30 \quad \lambda_2' = 2.11 \]

\[ \text{Sinh } \lambda_2 b = 4.064 \quad \text{Cosh } \lambda_2 b = 4.185 \quad \text{Coth } \lambda_2 b = 1.031 \]

\[ \text{Sin } \lambda_2 b = 0.857 \quad \text{Cos } \lambda_2 b = -0.515 \quad \text{cot } \lambda_2 b = -0.601 \]

\[ J_2 = \frac{1.69 \times 17.61}{1,056,000 \times 16.15} = 1.75 \times 10^{-6} \]

\[ K_2 = \frac{2.20 \times 16.91}{1,056,000 \times 16.15} = 2.19 \times 10^{-6} \]

\[ L_2 = \frac{1.30 \times 17.79}{1,056,000 \times 16.15} = 1.36 \times 10^{-6} \]

Finally,

\[ M_0 = -\frac{5.10 - 1.75}{4(4.36 + 2.18)} \quad P = -0.128 \ P \]

\[ B_1 = \frac{0.615 \times 3.35}{6.54} = 0.315 \]

\[ B_2 = \frac{1.30 \times 3.35}{6.54} = 0.666 \]

In member 2 the maximum pressure under the bolt will be

\[ P_{2 \text{ max.}} = 1,056,000 \left[ 1.36 \times 10^{-6} + \frac{1.75 \times 3.35 \times 10^{-6}}{2 \times 6.54} \right] P \]

\[ P_{2 \text{ max.}} = 1.91 \ P \]

Report No. 1951
which gives

\[ P = \frac{6,450}{2 \times 1.91} = 1,690 \text{ pounds.} \]

Find maximum moment in member 1. The positions of zero shear will occur for

\[ \cot \lambda_1 \left( x - \frac{a}{2} \right) \]

and

\[ \cosh \frac{\lambda_1 a}{2} \sin \frac{\lambda_1 a}{2} = 1 + 0.610 + 0.155 \]

The curve of figure 6 shows a solution exists for \( \theta > \pi \). Therefore \( \cosh \theta \sim 1 \) and finally \( \theta = 3.56 \), from which \( x = 7.60 \), which is physically impossible since the member is only 3.625 inches thick.

The maximum moment in member 1 must then occur at \( x = \frac{a}{2} = 1.812 \)

and is equal to

\[ M_1 \max. = \frac{P}{\Delta_1 \lambda_1} \sinh \frac{\lambda_1 a}{2} \sin \frac{\lambda_1 a}{2} = \frac{2 \times 0.128P}{\Delta_1} \left\{ \sinh \frac{\lambda_1 a}{2} \cos \frac{\lambda_1 a}{2} \right\} + \cosh \frac{\lambda_1 a}{2} \sin \frac{\lambda_1 a}{2} \]

\[ \sinh \frac{\lambda_1 a}{2} = 1.361 \quad \cosh \frac{\lambda_1 a}{2} = 1.689 \quad \sin \frac{\lambda_1 a}{2} = 0.898 \quad \cos \frac{\lambda_1 a}{2} = 0.440 \]

and

\[ M_1 \max. = \left[ \frac{1.361 \times 0.898 - 2 \times 0.128 (0.599 + 1.518)}{5.385 \times 0.615} \right] P \]

\[ M_1 \max. = 0.268 \quad \text{P} \]

which gives

\[ P = \frac{368}{0.268} = 1,370 \text{ pounds.} \]
Find maximum moment in member 2. The positions of zero shear will occur for

\[
\text{Coth } \lambda_2 (b - x) + \cot \lambda_2 (b - x) = \frac{2 + 0.666 (1.031 - 0.601)}{1.031 + 0.601 + 0.666} = 0.995
\]

and from figure 3

\[
\theta = 1.66 \text{ then } b - x = \frac{1.66}{1.30} = 1.275, \quad x = 0.350 \text{ inch.}
\]

Another solution of this equation exists at \( \theta = 4.729 \) when \( \text{Coth } \theta = 1.00 \). Then \( x = b - 3.63 \), which is physically impossible because the member is only 1.625 inches wide. Therefore, the maximum moment occurs in member 2 at \( x = 0.350 \) inch.

\[
\lambda_2 x = 0.455, \quad \lambda_2 (b - x) = 1.66
\]

\[
\text{Sinh } \lambda_2 x = 0.471 \quad \text{Cosh } \lambda_2 x = 1.105 \quad \text{sin } \lambda_2 x = 0.439 \quad \text{cos } \lambda_2 x = 0.898
\]

\[
\text{Sinh } \lambda_2 (b - x) = 2.534 \quad \text{Cosh } \lambda_2 (b - x) = 2.725 \quad \text{sin } \lambda_2 (b - x) = 0.996
\]

\[
\text{cos } \lambda_2 (b - x) = -0.091
\]

and

\[
M_{2 \text{ max.}} = -\frac{P}{31.55 \times 1.30} \left[ \begin{array}{c}
4.064 \times 0.439 \times 2.534 - 0.857 \times 0.471 \\
\times 0.996
\end{array} \right]
\]

\[
-\frac{0.128P}{31.55} \left[ \begin{array}{c}
4.064 (0.898 \times 2.53 + 0.439 \times 2.725) - 0.857 (-0.471 \\
\times 0.091 + 1.105 \times 0.996)
\end{array} \right]
\]

\[
M_{2 \text{ max.}} = -\frac{P}{6.50} \quad \text{or} \quad P = 6.50 \times 368 = 2,390 \text{ pounds.}
\]

Report No. 1951 -23-
Therefore the design load for the joint is controlled by the maximum moment in member 1 and the design load is 1,070 pounds.

The joint slip at this design load is

\[
\delta = 1.370 \left[ 8.91 + 1.36 - \frac{3.35}{2 \times 6.55} \right] \times 10^{-6} = 0.0129 \text{ inch.}
\]
Figure 1. --Notation used for joint of two members.
Figure 2. -- Curves for computation of $J_1$, $K_1$, and $L_1$ for joints of two members and $J_2$, $K_2$, and $L_2$ for joints of two or three members.
Figure 3. -- Curve for use in finding position of maximum moment
(Eqs. 21, 22, and 38).

Note:

For $\theta > \pi$

$\text{coth } \theta + \text{cot } \theta \approx 1 + \text{cot } \theta$
Figure 4. -- Notation used for joint of three members.
Figure 5. -- Curves for computation of $J_1$, $K_1$, and $L_1$ for joints of three members.
Figure 6. -- Curve for use in finding position of maximum moment (Eq. 37).

NOTE:
FOR $\theta > \pi$
\[
\frac{\cot \theta}{\coth \theta} \approx \cot \theta
\]
Figure 7. -- Shear, moment, and deflection diagrams for an eightpenny common nail under lateral loads in Douglas-sfir.