AN ABSTRACT OF THE THESIS OF

Steven R. Black for the degree of Masters of Science in Mathematics presented on November 26, 1996. Title: The Hawaiian Earring.

Redacted for Privacy

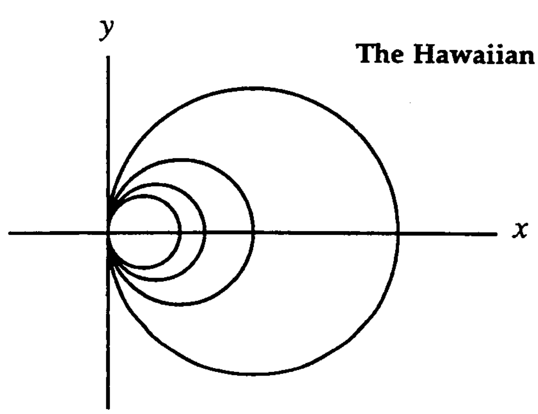
Abstract approved: ______________________________

William A. Bogley

The Hawaiian Earring is a topological subspace of the plane consisting of a countably infinite union of circles linked by a point of mutual tangency, and whose radii tend to zero. Precisely, for each positive integer n, put

$$C_n = \{(x,y) \in R^2 \mid (x - \frac{1}{n})^2 + y^2 = (\frac{1}{n})^2\}.$$

The Hawaiian Earring, HE, is the collection $\bigcup_{n \in Z^+} C_n$ together with the usual metric topology inherited from the plane.



The Hawaiian Earring

The temptation with the Hawaiian Earring is to compare it with the wedge of this same family of circles, the fundamental group of which is free over the class of elements represented by the implicit degree one maps. The reality, however, is that a contrast is more likely. Of key interest is the fact that the fundamental group of the Hawaiian Earring, with the origin taken as basepoint, is not free. The unveiling of this point came by way of separate enterprises, and inauspiciously. In the early 1950's, the group was first recognized as a subgroup of a projective system of free groups [1], but the imbedding was later found to be flawed. It would take some forty years for the identification to be salvaged [2]. With this, the conclusion

ABSTRACT (Continued)

that the fundamental group of the Hawaiian is not free awaited only citation for, in [3, 1952], G. Higman had demonstrated in a purely group theoretical treatise that this same subsystem was not free.

More recently, in a paper titled *The Fundamental Group of the Hawaiian Earring is not Free*, B. De Smit outlined a new proof of this fact [4]. Again, the treatment is intrinsically algebraic, invoking the aforementioned imbedding; and again, in our view, veritably inviting a topological recast. Indeed, this is the impetus and chief ambition of the paper you are now reading.

From a topological standpoint, the heart of the matter lies in the fact that the topology of the Hawaiian Earring is strictly coarser than that of the one-complex and, consequently, admits a wider class of loops. The salience of this point is not to be understated. With this in mind, there is some care here given to the topological subtleties of these spaces. The prescription of mapping properties, in particular, shall prove exigent to our topological presentation of De Smit's proof. In closing, the author wishes to extend his sincere gratitude to William A. Bogley for his generous collaboration on this project.

The Hawaiian Earring

by

Steven R. Black

A THESIS

submitted to

Oregon State University

in partial fulfillment of
the requirements for the
degree of

Master of Science

Presented November 26, 1996
Commencement June 1997

Master of Science thesis of Steven R. Black presented on November 26, 1996

APPROVED:

Redacted for Privacy

Major Professor, representing Mathematics

Redacted for Privacy

Chair of Department of Mathematics

Redacted for Privacy

Dean of Graduate School

I understand that my thesis will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my thesis to any reader upon request.

Redacted for Privacy

Steven R. Black, Author

TABLE OF CONTENTS

The Hawaiian Earring

Part 1. The Topologies of HE and HE_w

Let HE_w be the adjunction space obtained by wedging the C_n at $* = (0,0)$. The topology of this quotient is, in truth, the weak topology on $\bigcup\limits_{n\in Z^+} C_n$ with respect to the C_n; namely,

$$\tau_w = \{U \subseteq \bigcup_{n\in Z^+} C_n \mid U \cap C_n \overset{op}{\subseteq} C_n \quad \forall n \in Z^+\}.$$

It was indicated at the outset that this inherited topology is strictly finer than its parent on the Hawaiian Earring. Consequently, the natural correspondence $: HE_w \to HE$ is a continuous bijection that is not open. It is, in fact, a local homeomorphism except at $*$. This disparity in topology, while locallized, has a profound effect on many of the key invariants, such as first countability, compactness, and the fundamental group. We shall examine some of these matters in the following pages.

Also under consideration are functions on and into the Hawaiian Earring, continuity the issue. In particular, it will prove useful to formulate a continuity criterion for self maps of this space. We then close **Part 1** with an analysis of continuous loops in HE and HE_w. Here again, the local disjunction between topologies is evidenced in dramatic fashion.

1.1 Neighborhoods of the Origin

In light of the preceeding comments, it seems appropriate that this paper begin with a characterization of the topology at the distinguished point. In both spaces, the neighborhoods of $*$ are of the form $\bigcup\limits_{n\in Z^+} U_n$, where U_n is a neighborhood of $*$ in C_n. However, in HE, these neighborhoods must also satisfy $U_n = C_n$ for all but finitely many of the C_n. We demonstrate this explicitly.

Notation For each non-empty subcollection A of Z^+, let $HE(A)$ and $HE_w(A)$ be the set $\bigcup\limits_{a\in A} C_a$ together with the subspace topology inherited from HE and HE_w, respectfully.

Theorom 1.1 Let A be a non-empty subcollection of Z^+. Then each neighborhood of $*$ in $HE(A)$ contains all but finitely many of the C_a.

Proof Let U be any neighborhood of $*$ in $HE(A)$, and let $B_r(*) \cap HE(A)$ be a basic open neighborhood of $*$ contained in U. Putting $N = \min\{n \in Z^+ \mid 2/n < r\}$, we show that $C_a \subseteq U \;\; \forall a \geq N$. So let $(x,y) \in C_{a_0}$ for some $a_0 \geq N$. Then $(x - \frac{1}{a_0})^2 + y^2 = \frac{1}{a_0^2}$

$$\Rightarrow x^2 - \frac{2x}{a_0} + \frac{1}{a_0^2} + y^2 = \frac{1}{a_0^2} \;\;\Rightarrow x^2 + y^2 = \frac{2x}{a_0} = \frac{2xa_0}{a_0^2}.$$

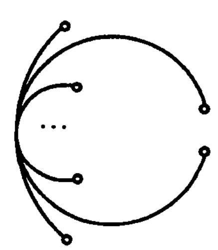
But, $x \leq \frac{2}{a_0} \;\Rightarrow\; xa_0 \leq 2 \;\Rightarrow\; 2xa_0 \leq 4$, so $x^2 + y^2 \leq \frac{4}{a_0^2} = (2/a_0)^2$, showing that $(x,y) \in \overline{B}_{\frac{2}{a_0}}(*) \subseteq \overline{B}_{\frac{2}{N}}(*) \subseteq B_r(*)$. In particular, $(x,y) \in B_r(*) \cap HE(A) \subseteq U$. √

Again, the neighborhoods of $*$ in HE_w need not contain all but finitely many of the C_n. In fact, they need contain none at all. For each $n \in Z^+$ and $b \in (0,1/2]$, let

$$C_n^b = \{\, (\frac{\cos 2\pi t + 1}{n}, \frac{\sin 2\pi t}{n}) \in R^2 \mid t \in [0,b) \cup (1-b,1] \,\}.$$

These sets are not open in HE_w for, if $m \neq n$, then $C_n^b \cap C_m = \{*\}$, which is not open in C_m. However, letting $\varepsilon : Z^+ \to (0,1/2]$ be any function, the union $\bigcup_{n \in Z^+} C_n^{\varepsilon(n)}$ is open in HE_w, and the collection

$$\mathcal{N}_* = \{\, \bigcup_{n \in Z^+} C_n^{\varepsilon(n)} \mid \varepsilon : Z^+ \to (0,1/2] \,\}$$



is a neighborhood basis at $*$ in HE_w, albeit uncountable. That one can distinguish between HE and HE_w is due wholly to the existence of such open sets, and variations thereof.

1.2 Distinguishing between HE and HE_w via Invariants

Having discussed the fact that the fundamental group of the Hawaiian Earring is not free, it has been acknowledged that HE is not homeomorphic to HE_w. However, weaker invariants may also detect the differences in these spaces. As cases in point, we demonstrate that HE_w is neither first-countable nor compact. Observe in both instances how open sets of the form pictured above compromise the invariant. Of course, HE is first-countable as a metrizable space, and compact as a closed and bounded subspace of the plane.

Theorom 1.2 HE_w is not first-countable.

Proof Let $\mathcal{V} = \{V_1, V_2, \ldots\}$ be any countable collection of neighborhoods of $*$ in HE_w. We construct a neighborhood of $*$ containing no V_i. For each $n \in Z^+$, pick $\varepsilon_n > 0$ so that the basic open neighborhood $B_{\varepsilon_n}(*) \cap C_n$ of $*$ in C_n is contained in $V_n \cap C_n \overset{op}{\subseteq} C_n$. Letting $B = \bigcup_{n \in Z^+} (B_{\frac{\varepsilon_n}{2}}(*) \cap C_n)$, we have for each $n \in Z^+$ that $B \cap C_n = B_{\frac{\varepsilon_n}{2}}(*) \cap C_n \overset{op}{\subseteq} C_n$, so B is a neighborhood of $*$ in HE_w. Moreover, for no positive integer n does B contain V_n; take x in the set $(C_n \cap B_{\varepsilon_n}(*)) - (C_n \cap B_{\frac{\varepsilon_n}{2}}(*))$. √

Theorom 1.3 HE_w is not compact.

Proof For each positive integer n, let $C_n^0 = C_n - \{(2/n, 0)\}$, and put $U_n = C_n \cup \bigcup_{m \neq n} C_m^0$. Then fixing n,

$$U_n \cap C_m = \begin{cases} C_m & \text{if } m = n \\ C_m^0 & \text{if } m \neq n. \end{cases}$$

In either case, this set is a member of the usual subspace topology on C_m, so U_n is open in HE_w. The collection $\{U_n\}_{n \in Z^+}$ is an open cover of HE_w with no finite subcover. √

1.3 Closed Sets and the Topology of $HE(A)$

In general, a space which is a union of finitely many closed subspaces has the weak topology with respect to those subspaces. We show next that for each subcollection A of Z^+, each C_a is closed in $HE(A)$. It is also true that, for reasons not intrinsic to the Hawaiian Earring, the subspace topology on $HE_w(A)$ is precisely the weak topology induced by the subspace topology on $HE(A)$. It follows that if A is finite, then $HE(A)$ is pointwise and topologically equivalent to $HE_w(A)$.

Theorom 1.4 **(Closed Sets)** Let A be any non-empty subcollection of Z^+.

(a) **i.** Each C_a is closed in $HE(A)$.

ii. Each C_a is closed in $HE_w(A)$.

Proof **i.** Consider for each $a \in A$ the function

$$f_a : R^2 \to R : (x,y) \to (x - \frac{1}{a})^2 + y^2.$$

Each is continuous and $C_a = f_a^{-1}(\{1/a^2\})$, where $\{1/a^2\} \overset{cl}{\subseteq} R$. So each C_a is closed in R^2 and, of course, bounded. Thus, each C_a is compact in R^2, so compact in $HE(A)$. Finally then, since $HE(A)$ is a Hausdorff space, each C_a is closed in $HE(A)$.

ii. It is often convenient to use the fact that a set is closed in $HE_w(A)$ if and only if its intersection with each C_a is closed in that C_a. Fixing $a \in A$,

$$C_a \cap C_b = \begin{cases} C_a & \text{if } b = a \\ \{*\} & \text{if } b \neq a\,. \end{cases}$$

In either case, the intersection is closed in C_a.

(b) **i.** $HE(A) \overset{cl}{\subseteq} HE$.

ii. $HE_w(A) \overset{cl}{\subseteq} HE_w$.

Proof **i.** ($\Rightarrow$) It suffices to show that each $C_n - \{*\}$ is open in HE, for then

$$HE - HE(A) = \bigcup_{n \notin A} (C_n - \{*\}) \overset{op}{\subseteq} HE,$$

which yields the desired result. So let $y \in C_k - \{*\}$ for some positive integer k. A simple but tedious geometric argument demonstrates that $d(\{y\}, C_{k+1}) \leq d(\{y\}, C_n)$ $\forall n \neq k$. Thus, since $\{y\}$ and C_{k+1} are closed, and $\{y\} \cap C_{k+1} = \varnothing$, $d(\{y\}, C_{k+1}) > 0$. Taking

$$\varepsilon = \frac{1}{2}\min\{d(\{y\}, C_{k+1}), d(\{y\}, \{*\})\},$$

the set $B_\varepsilon(y) \cap HE$ is a basic open neighborhood of y contained in $C_k - \{*\}$.

ii. For each positive integer n,

$$HE_w(A) \cap C_n = \begin{cases} C_n & \text{if } n \in A \\ \{*\} & \text{if } n \notin A. \end{cases}$$

In any case, the intersection is closed in C_n, so $HE_w(A) \overset{cl}{\subseteq} HE_w$. $\surd$

Denoting the complement of A in Z^+ by $\tilde{A}$, the opening comments of this section together with **part (b) i.** demonstrate that the Hawaiian Earring may be viewed as the wedge of its closed subspaces $HE(A)$ and $HE(\tilde{A})$. This can, at times, be helpful in determining whether a given function on the space is continuous. We discuss the pertinent mapping property in the following section.

1.4 Mappings : HE(A) $\to$ HE(B)

A key feature of a space with the weak topology is a strong universal mapping property. Namely, if X is a space having the weak topology with respect to a collection $\mathcal{A}$ of subspaces, then a function $: X \to Y^{sp}$ is continuous if and only if its resriction to each member of $\mathcal{A}$ is continuous. And it was discussed in section **1.3** that if A is a finite subcollection of Z^+, then $HE(A)$ has the weak topology with respect to $\{C_a \mid a \in A\}$. Thus, among the functions $: HE(A) \to HE(B)$, continuity of those for which A is finite is readily determined. However, when A is infinite, the matter becomes somewhat more delicate. Nevertheless, there are helpful formulations, and we detail one next. The result is then applied to constructing homeomorphisms and retractions on the Hawaiian Earring.

Notation For each positive integer n, let γ_n be the degree one map about C_n; namely,

$$\gamma_n : I \to C_n : t \to \left(\frac{-\cos 2\pi t + 1}{n}, \frac{\sin 2\pi t}{n}\right).$$

This prescription assumes no particular topology on C_n. For each n, we distinguish between γ_n in various codomains via context.

Theorom 1.5 Let A and B be non-empty subcollections of Z^+. Then a function $f: HE(A) \to HE(B)$ identifying $*$ is continuous if and only if

i. each restriction $f\big|_{C_a}: C_a \to HE(B)$ is continuous,

and **ii.** each neighborhood of $*$ in $HE(B)$ contains all but finitely many of the $f(C_a)$.

Proof ($\Rightarrow$) Assume f is continuous. Then each restriction $f\big|_{C_a}$ is continuous and, if A is finite, condition **(ii)** is met trivially. So assume A is infinite, and suppose by contradictory hypothesis that there exists a neighborhood U of $*$ in $HE(B)$ such that for each $N > 0$, there exists $a \in A$ with $a > N$ and $f(C_a) \not\subset U$. Then taking $N_1 > 0$, choose $a_1 > N_1$ and $x_1 \in C_{a_1}$ so that $f(x_1) \notin U$. Now take $a_2 > a_1$ and choose $x_2 \in C_{a_2}$ such that $f(x_2) \notin U$. Continuing in this fashion, we build a sequence (x_l) in $HE(A)$ converging to $*$ whose image sequence $(f(x_l))$ does not converge to $f(*) = *$ in $HE(B)$, contradicting the assumption that f is continuous.

($\Leftarrow$) Assume that conditions **(i)** and **(ii)** hold. We use adequacy of sequences to show that f is continuous at each point. So let $x \in HE(A)$, (x_l) a sequence in $HE(A)$ converging to x, and U a neighborhood of $f(x)$ in $HE(B)$.

case 1 Assume $x \in C_k - \{*\}$, and pick $\varepsilon > 0$ so that $B_\varepsilon(x) \cap C_a = \varnothing \ \ \forall a \in A - \{k\}$.

By convergence of (x_l), there exists $L_1 > 0$ such that $x_l \in B_\varepsilon(x) \cap HE(A) \overset{op}{\subseteq} HE(A) \ \ \forall l \geq L_1$.

Moreover, the subsequence $(x_l)_{l \geq L_1}$ is contained in C_k so, by continuity of $f\big|_{C_k}$, there exists $L_2 > L_1$ such that $f\big|_{C_k}(x_l) \in U \ \forall l \geq L_2$. Thus, for each $l \geq L_2$, $f(x_l) \in U$.

case 2 Assume $x = *$. By condition **(ii)**, the collection

$$\Lambda = \{a \in A \mid C_a \cap \{x_l \mid l \in Z^+\} \text{ and } f(C_a) \not\subset U\}$$

is finite. If Λ is empty, then f maps each x_l into U, and is continuous trivially. So suppose $\Lambda \neq \varnothing$ and, for each $a \in \Lambda$, let $\Lambda_a = \{x_l \mid x_l \in C_a\}$. If Λ_a is infinite, then its elements form a subsequence $(x_{l_{m(a)}})$ of (x_l) converging to x; so by continuity of $f\big|_{C_a}$, there exists $\lambda_a > 0$ so that

$$f\big|_{C_a}(x_{l_{m(a)}}) \in U \quad \forall l_{m(a)} \geq \lambda_a. \qquad \textbf{(1)}$$

For each $a \in \Lambda$, let

$$L_a = \begin{cases} \max\{l \in Z^+ \mid x_l \in \Lambda_a\} & \text{if } \Lambda_a \text{ is finite} \\ \lambda_a & \text{if } \Lambda_a \text{ is infinite}. \end{cases}$$

Putting $L = \max\{L_a \mid a \in \Lambda\}$, we show that $f(x_l) \in U \ \ \forall l > L$. So let $l_0 > L$ and $x_{l_0} \in C_{a_0}$.

subcase 1 If $a_0 \notin \Lambda$, then $f(x_{l_0}) \in f(C_{a_0}) \subseteq U$.

subcase 2 If $a_0 \in \Lambda$, then Λ_{a_0} must be infinite; for otherwise, $l_0 \le L_{a_0} \le L$. Thus

$$l_0 > \lambda_{a_0}, \text{so } f(x_{l_0}) = f\big|_{C_{a_0}}(x_{l_0}) \overset{(1)}{\in} U. \quad \surd$$

We will later invoke the following weaker version of sufficiency.

Theorem 1.6 **(A Sufficiency Condition)** Let A and B be infinite subcollections of Z^+, and order A naturally as $\{a_1, a_2, \ldots\}$ with $a_i < a_{i+1}$ for all i. Then a function $f: HE(A) \to HE(B)$ is continuous whenever

i. each restriction $f\big|_{C_a}: C_a \to HE(B)$ is continuous,

and **ii.** there exists a sequence of positive integers $(n_k) \to \infty$ such that

$$f(C_{a_k}) \subseteq \bigcup_{b \ge n_k} C_b \quad \forall k \in Z^+.$$

Proof We show that a function $f: HE(A) \to HE(B)$ satisfying condition **(ii)** identifies $*$ and meets the continuity criterion of **Theorem 1.5**; namely, each neighborhood of $*$ in $HE(B)$ contains all but finitely many of the $f(C_a)$.

To see that $f(*) = *$, suppose otherwise. Then $f(*) \in C_{b_0} - \{*\}$ for some $b_0 \in B$, and since $(n_k) \to \infty$, there exists $K > 0$ such that $n_k > b_0 \ \ \forall k \ge K$. Thus,

$$f(*) \in f(C_{a_k}) \subseteq \bigcup_{b \ge n_k} C_b \subseteq \bigcup_{b > b_0} C_b,$$

a contradiction since $(\bigcup_{b > b_0} C_b) \cap (C_{b_0} - \{*\}) = \varnothing$.

Now, to see that f satisfies condition **(ii)** of the continuity criterion, let U be any neighborhood of $*$ in $HE(B)$. By **Theorem 1.1**, U contains all but finitely many of the C_b; say $M = \max\{b \in B \mid C_b \not\subset U\}$. Because $(n_k) \to \infty$, there exists $J > 0$ such that $n_k > M \ \ \forall k \ge J$. Thus, for each such k,

$$f(C_{a_k}) \subseteq \bigcup_{b \ge n_k} C_b \subseteq \bigcup_{b > M} C_b \subseteq U. \quad \surd$$

Theorom 1.7 Let A and B be infinite subcollections of Z^+. Then $HE(A)$ is homeomorphic to $HE(B)$.

Proof Let f be any bijection between A and B. Because each degree one map $\gamma_{f(a)}: I \to C_{f(a)}$ respects the identifications of $\gamma_a : I \to C_a$, there exists for each $a \in A$ a unique continuous map $f_a : C_a \to C_{f(a)}$ such that $f_a \circ \gamma_a = \gamma_{f(a)}$.

$$\begin{tikzcd} & I \arrow[dl, "\gamma_a"'] \arrow[dr, "\gamma_{f(a)}"] & \\ C_a \arrow[rr, "f_a"'] & & C_{f(a)} \end{tikzcd}$$

And note that if $x \in C_{a_1} \cap C_{a_2}$ for some $a_1 \neq a_2$, then $x = *$, so

$$f_{a_1}(x) = (\gamma_{f(a_1)} \circ \gamma_{a_1}^{-1})(x) = (\gamma_{f(a_2)} \circ \gamma_{a_2}^{-1})(x) = f_{a_2}(x).$$

Therefore, there exists a function $\tilde{f} : HE(A) \to HE(B)$ such that $\tilde{f}\big|_{C_a} = f_a \quad \forall a \in A$. We show that $\tilde{f}$ is a homeomorphism.

injective Assume $\tilde{f}(x) = \tilde{f}(y)$ for some $x, y \in HE(A)$, and let $x \in C_{a_1}$ and $y \in C_{a_2}$. Then

$$\tilde{f}\big|_{C_{a_1}}(x) = \tilde{f}(x) = \tilde{f}(y) = \tilde{f}\big|_{C_{a_2}}(y).$$

Thus, if $a_1 = a_2$, then $x = y$ by injectivity of $\tilde{f}\big|_{C_{a_1}}$. And if $a_1 \neq a_2$, then by injectivity of f, $C_{f(a_1)} \cap C_{f(a_2)} = \{*\}$, so $x = y = *$.

surjective Let $x \in C_b$ for some $b \in B$. Then by surjectivity of f, there exists $a \in A$ such that $f(a) = b$, so $x \in C_{f(a)}$. And by surjectivity of $\tilde{f}\big|_{C_a}$, there exists $y \in C_a$ such that $\tilde{f}\big|_{C_a}(y) = x$, so $\tilde{f}(y) = x$.

To see that $\tilde{f}$ is continuous, note first that each restriction $\tilde{f}\big|_{C_a} = \gamma_{f(a)} \circ \gamma_a^{-1}$ is continuous as the composition of continuous functions. And recall that by **Theorom 1.1**, each neighborhood of $*$ in $HE(B)$ contains all but finitely many of the C_b. Thus, since f is injective, each such neighborhood contains all but finitely many of the $C_{f(a)} = \tilde{f}(C_a)$. Therefore, $\tilde{f}$ is continuous by **Theorom 1.5**. The same argument shows that f^{-1} is continuous.

We need only show that $(\tilde{f})^{-1} = f^{-1}$. So let $x \in C_b$ for some $b \in B$. Then by surjectivity of f, there exists $a \in A$ such that $f(a) = b$, so

$$(\tilde{f})^{-1}(x) = (\tilde{f}|_{C_a})^{-1}(x) = (\gamma_a \circ \gamma_{f(a)}^{-1})(x) = (\gamma_{f^{-1}(b)} \circ \gamma_{f(a)}^{-1})(x) = f^{-1}|_{C_b}(x) = f^{-1}(x). \quad \surd$$

C 1.8 For each non-empty subcollection A of Z^+, $HE(A)$ is a retract of HE.

Proof Define the function

$$\rho : HE \to HE(A) : x \to \rho(x) = \begin{cases} x & \text{if } x \in HE(A) \\ * & \text{otherwise} \end{cases} .$$

Each restriction $\rho|_{C_n}$ is continuous, either as the identity function if $n \in A$, or the constant function if $n \notin A$. And since each neighborhood of $*$ in $HE(A)$ contains all but finitely many of the C_a, each such neighborhood obviously contains all but finitely many of the $\rho(C_n)$. So ρ is continuous by **Theorom 1.5** and, therefore, a retraction. $\surd$

One might note that the same is true for HE_w; that is, it has each $HE_w(A)$ as a retract. The analagous maps suffice. That each is continuous is a trivial consequence of the universal mapping property for the weak topology.

1.5 Loops in HE and HE$_w$

It is a general fact of one-complexes that compact subspaces meet only finitely many vertices and one-cells. Consequently, each continuous loop in HE_w meets only finitely many of the C_n away from $*$. This fails spectacularly in the Hawaiian Earring, as its coarser topology admits loops of exceeding complexity, including those that cover infinitely many of the C_n. We shall construct some of these loops in this section. But first, let us prove compact supports for HE_w directly.

Theorom 1.9 (**Compact Supports**) For each loop λ in HE_w, there exists a finite subcollection A of Z^+ such that $im(\lambda) \subseteq HE_w(A)$.

Proof Let $\lambda : I \to HE_w$ be a loop in HE_w based at $*$ and, for each positive integer n in the collection,

$$\Lambda = \{n \in Z^+ \mid im(\lambda) \cap (C_n - \{*\}) \neq \varnothing\},$$

select one member x_n of $C_n \cap (im(\lambda) - \{*\})$. It suffices to show that the set $C = \{x_n \mid n \in \Lambda\}$ is finite. We accomplish this via the general topological fact that closed, discrete subspaces of compact spaces are finite. Observe first that C is contained in $im(\lambda)$, which is compact as the continuous image of the compact set I. For each positive integer m not in Λ, select $y_m \in C_m - \{*\}$, and put

$$U = \bigcup_{n \in \Lambda}(C_n - \{x_n\}) \cup \bigcup_{m \notin \Lambda}(C_m - \{y_m\}).$$

Then for each positive integer l,

$$U \cap C_l = \begin{cases} C_l - \{x_l\} & \text{if } l \in \Lambda \\ C_l - \{y_l\} & \text{if } l \notin \Lambda. \end{cases}$$

In each case, $U \cap C_l \overset{op}{\subseteq} C_l$, so $U \overset{op}{\subseteq} HE_w$. Thus,

$$im(\lambda) - C = U \cap im(\lambda) \overset{op}{\subseteq} im(\lambda) \Rightarrow C \overset{cl}{\subseteq} im(\lambda).$$

Now, for each positive integer n, observe that $C_n - \{*\} \overset{op}{\subseteq} HE_w$. In particular, for each $n \in \Lambda$,

$$\{x_n\} = C \cap (C_n - \{*\}) \overset{op}{\subseteq} C,$$

so C is discrete. $\surd$

Next, we construct a prototype for a large class of loops in the Hawaiian Earring. For each $k \in Z^+$ and $n \in Z$, define a corespondence $h_{k,n} : [\frac{1}{k+1}, \frac{1}{k}] \to I$ by

$$h_{k,n}(\frac{1}{k+t}) = \begin{cases} 1-t & \text{if } n > 0 \\ t & \text{if } n < 0 \\ 0 & \text{if } n = 0 \end{cases} \qquad 0 \leq t \leq 1.$$

One observes that each $h_{k,n}$ is continuous; in fact, a homeomorphism. Now, given a sequence of integers $\sigma = (n_k)$, consider for each k the composition

$$[\frac{1}{k+1}, \frac{1}{k}] \xrightarrow{h_{k,n_k}} I \xrightarrow{\gamma_{|n_k|}} C_{|n_k|}.$$

If $x \in [\frac{1}{j+1}, \frac{1}{j}] \cap [\frac{1}{k+1}, \frac{1}{k}]$ with $j \neq k$, then x is an endpoint of one of these intervals, so

$$(h_{j,n_j} \circ \gamma_{|n_j|})(x) = (h_{k,n_k} \circ \gamma_{|n_k|})(x) = *.$$

Therefore, there exists a function $q_\sigma : I \to HE$ defined by

$$\textbf{i. } q_\sigma\Big|_{[\frac{1}{k+1},\frac{1}{k}]} = \gamma_{|n_k|} \circ h_{k,n_k} \quad \text{for each } k \in Z^+,$$

and $$\textbf{ii. } q_\sigma(0) = * .$$

In words, q_σ either wraps each interval $[\frac{1}{k+1},\frac{1}{k}]$ about $C_{|n_k|}$, or identifies the interval to $*$, depending on the value of the corresponding sequence element n_k. If $n_k > 0$, the interval (from $\frac{1}{k+1}$ to $\frac{1}{k}$) is wrapped clockwise about $C_{|n_k|}$; counterclockwise if $n_k < 0$. In the case that $n_k = 0$, the interval is mapped to $\{*\}$. Again, each element of the collection $\{0\} \cup \{1/k \mid k \in Z^+\}$ is assigned to $*$, thus assuring a well defined function.

We now consider the class of those sequences of integers which induce loops in the Hawaian Earring. At stake is continuity.

Theorom 1.10 Given a sequence of integers $\sigma = (n_k)$, the induced function $q_\sigma : I \to HE$ is continuous if and only if

i. $(n_k) \to 0$,

or **ii.** the non-zero members of (n_k) form a subsequence (n_j) such that $(|n_j|) \to \infty$.

Proof Let $\sigma = (n_k)$ be any sequence of integers.

($\Leftarrow$) Under each of the conditions (**i**) and (**ii**), the proof that q_σ is continuous over $(0,1]$ proceeds the same. If $x \in (0,1]$, let V be a neighborhood of x in I intersecting at most finitely many of the $I_k = [\frac{1}{k+1},\frac{1}{k}]$, say $I_{k_1},\dots,I_{k_l}$. Then $V \subseteq \bigcup_{i=1}^{l} I_{k_i}$ and, by the pasting lemma, the restriction of q_σ to this finite union is also continuous. In particular, q_σ is continuous at x.

We use adequacy of sequences to show that q_σ is continuous at $x = 0$. So let (x_l) be a sequence in I converging to zero, and let U be a neighborhood of $q_\sigma(0) = *$ in HE.

i. Assume $(n_k) \to 0$. Then $\exists K > 0$ such that $n_k = 0 \ \forall k \geq K$. And by convergence of (x_l), $\exists L > 0$ such that $x_l \in B_{\frac{1}{2K}}(0) \cap I \overset{op}{\subseteq} I \ \ \forall l \geq L$. So let $l_0 \geq L$. Then

$$x_{l_0} \in [\frac{1}{k_0+1}, \frac{1}{k_0}] \text{ for some } k_0 \geq 2K > K,$$

so $n_{k_0} = 0$. Thus,

$$q_\sigma(x_{l_0}) \in q_\sigma([\frac{1}{k_0+1}, \frac{1}{k_0}]) = * \in U,$$

showing that q_σ is continuous at $x = 0$.

ii. Assume that the non-zero members of (n_k) form a subsequence (n_j) satisfying $(|n_j|) \to \infty$. By **C 1.1**, the neighborhood U of $*$ contains nearly all of the C_n, so there exists a positive integer N such that $C_n \subseteq U \ \forall n \geq N$. And by assumption, $\exists J > 0$ such that $|n_j| \geq N \ \forall j \geq J$. Further, there exists by convergence of (x_l) an $L > 0$ such that

$$x_l \in B_{\frac{1}{2J}}(0) \cap I \overset{op}{\subseteq} I \quad \forall l \geq L.$$

So let $l_0 \geq L$. Then

$$x_{l_0} \in [\frac{1}{k_0+1}, \frac{1}{k_0}] \text{ for some } k_0 \geq 2J > J.$$

case 1 If $n_{k_0} = 0$, then $q_\sigma(x_{l_0}) = * \in U$.

case 2 If $n_{k_0} \neq 0$, then since $k_0 \geq J$, $|n_{k_0}| \geq N$, whence

$$q_\sigma(x_{l_0}) \in q_\sigma([\frac{1}{k_0+1}, \frac{1}{k_0}]) = C_{|n_{k_0}|} \subseteq U.$$

($\Rightarrow$) Assume q_σ is continuous, and let $N \geq 0$. For each positive integer k, denote the midpoint of the interval $[\frac{1}{k+1}, \frac{1}{k}]$ by m_k. Because the sequence (m_k) of midpoints converges to zero in I, the image sequence $(q_\sigma(m_k))$ converges by assumption to $q_\sigma(0) = *$ in HE. Thus, there exists a positive integer K such that

$$q_\sigma(m_k) \in B_{\frac{2}{N}}(*) \cap HE \overset{op}{\subseteq} HE \quad \forall k \geq K. \tag{1}$$

Note too that

$$q_\sigma(m_k) = \begin{cases} * & \text{if } n_k = 0 \\ (\frac{2}{|n_k|}, 0) & \text{otherwise}. \end{cases}$$

So suppose $\sigma = (n_k)$ does not converge to zero, and consider the subsequence (n_j) of non-zero members of (n_k). By **(1)**,

$$q_\sigma(m_j) = (\frac{2}{|n_j|}, 0) \in B_{\frac{2}{N}}(*) \quad \forall j \geq k.$$

Therefore, $|n_j| > N \quad \forall j \geq K$, showing that $(|n_j|) \to \infty$. √

Part 2. The Fundamental Groups of HE and HE_w

We now have prescriptions for a large class of loops in $\bigcup_{n\in Z^+} C_n$ which are continuous in HE but not in HE_w. Rest assured that this collection is far from complete, even up to path homotopy. However, those in hand are sufficient to show that the fundamental groups of these spaces are very different. As we shall see, their existence alone renders the fundamental group of the Hawaiian Earring uncountable.

In keeping with our compare and contrast treatment of these spaces, it should be observed that their fundamental groups do evidence commonalities. Notably, they share many of the same subgroups. In section 2.3, we shall digress for a moment to analyze some interesting mapping properties of these groups. Our chief goal, the proof that the fundamental group of the Hawaiian Earring is not free, is taken up in the final section.

Notation Let $G = \pi_1(HE,*)$ and $G_w = \pi_1(HE_w,*)$, and for each non-empty subcollection A of Z^+, let $G(A) = \pi_1(HE(A),*)$ and $G_w(A) = \pi_1(HE_w(A),*)$. When viewing the degree one maps $\gamma_n : I \to C_n$ as loops in HE, we put $x_n = [\gamma_n] \in G$.

2.1 The Natural Imbedding : $\mathrm{G_w} \to \mathrm{G}$

The fundamental group of the wedge HE_w, according to Van Kampen's theorom, is free over the basis of path classes represented by the degree one maps about the C_n. So what obstructs the fundamental group of the Hawaiian Earring from being free? Might it be so simple as the existence of non-trivial relations among the basic group elements just mentioned? It is shown next that the homomorphism : $G_w \to G$ induced by the natural correspondence is injective, ruling out such rudimentary explanations.

Theorom 2.1 The homomorphism $f_\# : G_w \to G$ induced by the identity function is injective, but not surjective.

Proof To see that $f_\#$ is injective, assume $[\lambda] \in \ker(f_\#)$, and let $A = \{ n \in Z^+ \mid im(\lambda) \cap (C_n - \{*\}) \neq \varnothing \}$. By compact supports, A is finite, so the set $\bigcup_{n\in A} C_n$ inherits the same subspace topology from HE and HE_w. Letting $\rho : HE \to HE(A)$ be the usual retraction, put $\rho' = \rho \circ f$.

$$\begin{tikzcd} I \arrow[r, "\lambda"] & HE_w \arrow[r, "f"] \arrow[d, "\rho'"'] & HE \arrow[dl, "\rho"] \\ & HE_w(A) = HE(A) & \end{tikzcd}$$

By assumption,

$$f_\#([\lambda]) = [*]_G \Rightarrow (\rho_\# \circ f_\#)([\lambda]) = [*]_{G_w(A)} \Rightarrow \rho'_\#([\lambda]) = [*]_{G_w(A)}.$$

Thus, $[\lambda] = (\iota'_\# \circ \rho'_\#)([\lambda]) = \iota'_\#([*]_{G_w(A)}) = [*]_{G_w}$.

To see that $f_\#$ is not surjective, consider the sequence $\sigma = 1,2,3,...$ of positive integers. Because σ tends to infinity, the induced function $q_{\sigma_m} : I \to HE$ is continuous by **Theorom 1.5** and, therefore, represents a path class in G. Just suppose there were $[\mu] \in G_w$ such that $f_\#([\mu]) = [q_\sigma]$. By compact supports, there exists a positive integer b such that $im(\mu) \cap (C_b - \{*\}) = \varnothing$. Noting that C_b inherits the same subspace topology from HE and HE_w, let $\rho : HE \to C_b$ be the usual retraction.

$$\begin{tikzcd} I \arrow[r, "q_\sigma"] \arrow[d, "\mu"'] & HE \arrow[d, "\rho"] \\ HE_w \arrow[ur, "f"] & C_b \end{tikzcd}$$

By assumption,

$$f_\#([\mu]) = [q_\sigma] \Rightarrow \rho_\#([q_\sigma]) = (\rho_\# \circ f_\#)([\mu]) = [*]_{G(\{b\})},$$

a contradiction since $\rho \circ q_\sigma$ is essential in $HE(\{b\})$. $\surd$

Of course, the fact that the inclusion $: G_w \to G$ is not surjective does not prove that G and G_w are not isomorphic. It is the degree to which this function fails to cover G that does the trick.

Theorom 2.2 G is uncountable.

Proof For each positive integer n, let $\rho_n : G \to G(\{n\})$ be the usual retraction induced homomorphism, $\pi_n : \prod_{i=1}^{\infty} Z \to Z$ the projection onto the n'th coordinate, and $\psi_n : G(\{n\}) \to Z$ the standard isomorphism. By the universal mapping property for products, there exists a unique homomorphism $\varphi : G \to \prod_{i=1}^{\infty} Z$ such that for each $n \in Z^{+}$, $\pi_n \circ \varphi = \psi_n \circ \rho_n$.

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & \prod_{i=1}^{\infty} Z \\ \rho_n \downarrow & & \downarrow \pi_n \\ G(\{n\}) & \xrightarrow[\psi_n]{\cong} & Z \end{array}$$

We show that φ maps G surjectively onto the uncountable product $\prod_{i=1}^{\infty} Z$.

So let $m = (m_1, m_2, \ldots) \in \prod_{i=1}^{\infty} Z$, and consider the sequence $\sigma_m = \underbrace{1,\ldots,1}_{|m_1|}, \underbrace{2,\ldots,2}_{|m_2|}, \ldots$.

Again, the function q_{σ_m} induced by this sequence forms a continuous loop in HE. For each positive integer n,

$$(\psi_n \circ \rho_n)([q_{\sigma_m}]) = \psi_n([\gamma_n]^{|m_n|}) = |m_n|,$$

so by the UMP, $(\pi_n \circ \varphi)([q_{\sigma_m}]) = |m_n|$. Therefore, $\varphi([q_{\sigma_m}]) = m$. √

2.2 Subgroups of $\mathbf{G}$ and $\mathbf{G_w}$

Given a non-empty subcollection A of Z^{+}, we show next that the inclusion $: G(A) \to G$ is injective, and so may be used to view $G(A)$ as a subgroup of G. And it was shown previously that if A is finite, then the metric topology on $HE(A)$ is identically that of the wedge $HE_w(A)$. It follows from the Van Kampen theorom that G contains non-abelian free subgroups. We begin with a lemma.

Lemma 2.3 Let A and B be non-empty subcollections of Z^{+} satisfying $A \cap B = \varnothing$, and let $\iota : HE(A) \to HE$ be the inclusion, and $\rho : HE \to HE(B)$ the usual retraction. Then $im(\iota_{\#}) \subseteq \ker(\rho_{\#})$.

Proof Let $[\lambda] \in G(A)$. Then $im(\iota \circ \lambda) \subseteq HE(A) \subseteq HE(\tilde{B})$, so $\rho(im(\iota \circ \lambda)) = \{*\}$ and, therefore,

$$\rho_{\#}(\iota_{\#}([\lambda])) = [\rho \circ \iota \circ \lambda] = [*]_{G(B)}. \quad √$$

Theorom 2.4 Let A be a non-empty subcollection of Z^+, and let $\iota : HE(A) \to HE$ be the inclusion, and $\rho : HE \to HE(A)$ the standard retraction.

(a) $\iota_\# : G(A) \to G$ is injective, and $\rho_\# : G \to G(A)$ is surjective.

Proof The composition $\rho \circ \iota = 1_{HE(A)}$, so by functorality,

$$\rho_\# \circ \iota_\# = (\rho \circ \iota)_\# = 1_{G(A)}. \quad \surd$$

(b) If A is contained properly in Z^+, then $\iota_\#$ is not surjective, and $\rho_\#$ is not injective.

Proof Let $b \in Z^+ - A$, and suppose that there were a loop $\lambda : I \to HE(A)$ based at $*$ such that $\iota_\#([\lambda]) = [\gamma_b]$, where γ_b is the degree one map about C_b in HE.

$$\begin{array}{ccc} HE(A) & \xrightarrow{\iota} & HE \\ \lambda \uparrow & & \downarrow \rho_b \\ I & \xrightarrow[\gamma_b]{} & C_b \end{array}$$

Then, $(\rho_{b\#} \circ \iota_\#)([\lambda]) = \rho_{b\#}([\gamma_b]) \neq [*]_{G(B)}$. However, since $\{b\} \cap A = \varnothing$, we have by the preceeding lemma that, $(\rho_{b\#} \circ \iota_\#)([\lambda]) = [*]_{G(B)}$, a contradiction.

To see that $\rho_\#$ is not injective, let $\iota_b : C_b \to HE$ be the inclusion, and observe that $\iota_b \circ \gamma_b$ is essential in HE.

$$I \xrightarrow{\gamma_b} C_b \xrightarrow{\iota_b} HE \xrightarrow{\rho} HE(A)$$

Again by the preceeding lemma, $\rho_\#(\iota_{b\#}([\gamma_b])) = [*]_{G(A)}$, demonstrating that $\ker(\rho_\#)$ is non-trivial. $\surd$

2.3 Endomorphisms of G and G_w

While much of this paper has been tempered toward the realization of the fact that G is not free, this section is advanced soley out of an appreciation for the differences *and* similarities of G and G_w. There are two results, both involving endomorphisms of these groups.

Definition A group H is called ***Hopfian*** if each surjective homomorphism $: H \to H$ is injective; ***co-Hopfian*** if each injective homomorphism $: H \to H$ is surjective.

Theorom 2.5 G and G_w are neither Hopfian nor co-Hopfian.

Proof Let A be a proper infinite subcollection of Z^+. Then by **Theorom 1.7,** there exists a homeomorphism F from $HE(A)$ onto HE. So consider the map

$$F_\# \circ \rho_\# : G \to G(A) \to G.$$

Induced by a homeomorphism, $F_\#$ is surjective and, by **Theorom 2.4 a**, so too is $\rho_\#$. Thus, the composition $F_\# \circ \rho_\#$ is surjective. However, since A is contained properly in Z^+, $\rho_\#$ is not injective, and so neither is $F_\# \circ \rho_\#$. Hence, G is not Hopfian. And because the inclusion $\iota_\# : G_w(A) \to G$ is not surjective, a similar argument shows that the composite homomorphism

$$\iota_\# \circ F_\#^{-1} : G \to G(A) \to G$$

is injective but not surjective.

By the universal mapping property for free groups, any function $: G_w \to X^{sp}$ is uniquely determined by its assignment of the generators. So viewing the degree one maps γ_n as loops in HE_w, put $w_n = [\gamma_n] \in G_w$, and define functions $f, g : G_w \to G_w$ by

$$f(w_n) = \begin{cases} w_1 & \text{if } n < 3 \\ w_{n-1} & \text{if } n \geq 3 \end{cases}, \quad \text{and} \quad g(w_n) = w_{n+1}.$$

It is easy to see that f is a surjective homomorphism that is not injective, and g an injective homomorphism that is not surjective. √

Theorom 2.6 Given $i \in Z^+$, the inner automorphism

$$\varphi_i : G \to G : y \to x_i y x_i^{-1},$$

is not induced by a continuous function $: (HE, *) \to (HE, *)$. However, the analogous automorphisms for G_w *are* induced by pointed maps.

Proof Suppose to the contrary that $f : (HE, *) \to (HE, *)$ is a continuous function such that $f_\# = \varphi$. Then for each positive integer n, $f_\#(x_n) = x_i x_n x_i^{-1}$. And note that each loop belonging to the path class $x_i x_n x_i^{-1}$ must touch the outer most point of C_i, $(2/i, 0)$. In particular, $(2/i, 0) \in im(f \circ \gamma_n)$. Thus, for each positive integer n, there exists

$b_n \in im(\gamma_n) = C_n$ such that $f(b_n) = (2/i, 0)$. But then, the b_n form a sequence in HE converging to $*$ whose image sequence $(f(b_n))$ does not converge to $f(*) = *$, contradicting the fact that f is continuous.

Now, view φ_i as the inner automormphism $: G_w \to G_w : y \to w_i y w_i^{-1}$, where $w_i = [\gamma_i] \in G_w$. For each positive integer j, divide the unit interval into thirds and form the concatenation $\gamma_j \gamma_i \gamma_j^{-1} : I \to HE_w$. Because each loop $\gamma_j \gamma_i \gamma_j^{-1}$ respects the identifications of $\gamma_j : I \to C_j$, there exists for each j a unique continuous map $g_j : C_j \to HE_w$ such that $g_j \circ \gamma_j = \gamma_j \gamma_i \gamma_j^{-1}$.

$$\begin{tikzcd} & I \arrow[dl, "\gamma_j"'] \arrow[dr, "\gamma_i\gamma_j\gamma_i^{-1}"] & \\ C_j \arrow[rr, "f_j"'] & & HE_w \end{tikzcd}$$

If $x \in C_j \cap C_{j'}$ for some $j \neq j'$, then $x = *$, so

$$g_j(x) = (g_j \circ \gamma_j)(0) = \gamma_j \gamma_i \gamma_j^{-1}(0) = \gamma_{j'} \gamma_i \gamma_{j'}^{-1}(0) = (g_{j'} \circ \gamma_{j'})(0) = g_{j'}(x).$$

Therefore, there exists a function $g : (HE_w, *) \to (HE_w, *)$ such that $g\big|_{C_j} = g_j \ \ \forall j$. So let $y \in G_w$. We may then write $y = \varepsilon(1)^{a_1} \cdots \varepsilon(s)^{a_s}$, where each a_k is a non-zero integer and ε a map $: \{1, \ldots, s\} \to \{w_n \mid n \in Z^+\}$ with $\varepsilon(k) \neq \varepsilon(k+1)$ for any k. And finally,

$$g_\#(y) = g_\#(\varepsilon(1)^{a_1}) \cdots g_\#(\varepsilon(s)^{a_s}) = w_i \varepsilon(1)^{a_1} w_i^{-1} \cdots w_i \varepsilon(s)^{a_s} w_i^{-1} = w_i y w_i^{-1},$$

so $g_\# = \varphi_i$. $\surd$

2.4 G is not Free

As noted in the introduction, it was established in the middle of this century that the fundamental group of the Hawaiian Earring is not free. Some forty years later, B. De Smit [1] suggested an alternative proof. Both versions are purely algebraic, and involve imbedding the group in a projective system of free groups. In this section, we proffer a topological recast of De Smit's 1992 account.

Note first that an uncountable free group Φ is free on uncountably many generators and, therefore, $Hom(\Phi, Z)$ is uncountable. For let $\mathcal{B}$ be an uncountable basis for Φ. Then for each

$b \in \mathcal{B}$, the correspondence

$$\psi_b : \mathcal{B} \to Z : x \to \psi_b(x) = \begin{cases} 1 & \text{if } x = b \\ 0 & \text{otherwise} \end{cases}$$

extends to a homomorphism $\varphi_b : \Phi \to Z$. Since $\varphi_b \neq \varphi_{b'}$ for $b \neq b'$, it follows that $Hom(\Phi, Z)$ is uncountable. Thus, having demonstrated in **Theorom 2.2** that G is uncountable, it suffices to show that $Hom(G,Z)$ is countable. To this end, define a correspondence,

$$\psi : Hom(G,Z) \to \prod_{i=1}^{\infty} Z_i : \varphi \to (\varphi(x_1), \varphi(x_2), \ldots).$$

We show that ψ maps the group $Hom(G,Z)$ injectively to $\bigoplus_{i=1}^{\infty} Z_i$, which is countable. In the first phase, showing that the image of ψ is contained in the direct sum, the proof requires the existence of special loops in the Hawaiian Earring. At the group level, we call the elements of G represented by these loops the **L-set**, and there is a subsection devoted to their construction.

Lemma 2.7 For each positive integer j, let y_j be an element of G supported in $\bigcup_{n \geq n_j} C_n$, where (n_j) is a sequence of positive integers tending toward infinity. Then there exists an endomorphism of G assigning each x_j to y_j.

Proof For each positive integer j, denote $y_j = [\omega_j]$, where $\omega_j : I \to HE$. Since each loop ω_j respects the identifications of the degree one map $\gamma_j : I \to C_j$, there exists for each j a unique continuous map $\varphi_j : C_j \to HE$ such that $\varphi_j \circ \gamma_j = \omega_j$.

$$\begin{tikzcd} I \arrow[r, "\omega_j"] \arrow[d, "\gamma_j"'] & HE \\ C_j \arrow[r, "\iota_j"'] \arrow[ur, "\varphi_j"] & HE \arrow[u, "\varphi"'] \end{tikzcd}$$

If $x \in C_j \cap C_{j'}$ for some $j \neq j'$, then $x = *$, so

$$\varphi_j(x) = (\varphi_j \circ \gamma_j)(0) = \omega_j(0) = \omega_{j'}(0) = (\varphi_{j'} \circ \gamma_{j'})(0) = \varphi_{j'}(x).$$

Therefore, there exists a function $\varphi : HE \to HE$ such that $\varphi|_{C_j} = \varphi_j \;\; \forall j$. And for each positive integer j,

$$\varphi(C_j) = \varphi_j(C_j) = (\varphi_j \circ \gamma_j)(I) = \omega_j(I) \subseteq \bigcup_{n \geq n_j} C_n,$$

where by assumption, $(n_j) \to \infty$. Thus, φ is continuous by the sufficiency condition of **Theorom 1.6**, so induces a homomorphism $\varphi_{\#} : G \to G$. And finally,

$$\varphi_{\#}(x_j) = \varphi_{\#}([\iota_j \circ \gamma_j]) = \varphi_{j\#}([\gamma_j]) = [\varphi_j \circ \gamma_j] = [\omega_j] = y_j. \quad \surd$$

We next describe a class of elements from G called the L-set, in recognition of Lenstra's contribution [4]. Let $\sigma = (m_k)$ be a sequence of positive integers, and for each $l \in Z^+$, define a sequence $(\lambda_n^{(l)})$ of loops in HE by

$$\lambda_n^{(l)} = \begin{cases} \text{the constant loop } c_* : I \to HE : x \to * & \text{if } n < l \\ \text{the degree one map } \gamma_l : I \to C_l & \text{if } n = l \\ \gamma_l(\gamma_{l+1}(\cdots(\gamma_{n-1}\gamma_n^{m_{n-1}})^{m_{n-2}})\cdots)^{m_{l+1}})^{m_l} & \text{if } n > l \end{cases} .$$

For example,

$$(\lambda_n^{(1)}) = \gamma_1, \gamma_1\gamma_2^{m_1}, \gamma_1(\gamma_2\gamma_3^{m_2})^{m_1}, \gamma_1(\gamma_2(\gamma_3\gamma_4^{m_3})^{m_2})^{m_1}, \ldots ,$$

$$(\lambda_n^{(2)}) = c_*, \gamma_2, \gamma_2\gamma_3^{m_2}, \gamma_2(\gamma_3\gamma_4^{m_3})^{m_2}, \gamma_2(\gamma_3(\gamma_4\gamma_5^{m_4})^{m_3})^{m_2}, \ldots ,$$

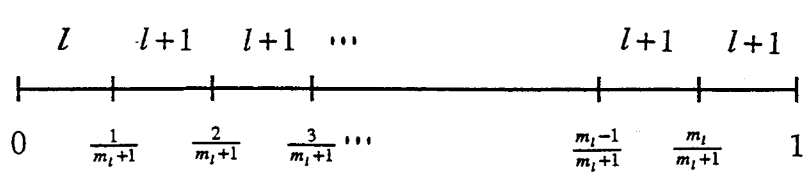
$$\vdots$$

$$\text{and } (\lambda_n^{(l)}) = \underbrace{c_*, \ldots, c_*}_{l-1 \text{ times}}, \gamma_l, \gamma_l\gamma_{l+1}^{m_l}, \gamma_l(\gamma_{l+1}, \gamma_{l+2}^{m_{l+1}})^{m_l}, \gamma_l(\gamma_{l+1}(\gamma_{l+2}\gamma_{l+3}^{m_{l+2}})^{m_{l+1}})^{m_l}, \ldots .$$

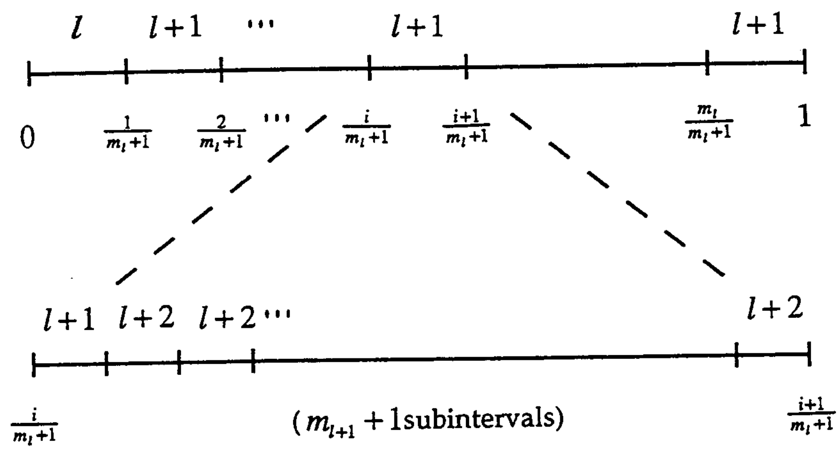
Because composition of paths is associative only up to path-homotopy, these prescriptions will need to be made more precise. So fix a sequence $(\lambda_n^{(l)})$. When $n \le l$, the loop $\lambda_n^{(l)}$ is unambiguous, being either the constant loop c_*, or the degree one map γ_n. However, when $n > l$, $\lambda_n^{(l)}$ is a composition of loops, the concatenation of which we fashion with particularity.

Fixing $n > l$, the issue here lies in specifying the rate at which $\lambda_n^{(l)}$ traces out the images of $\gamma_l, \gamma_{l+1}, \ldots, \gamma_n$. This we do by specifying a subdivision of the unit interval into finitely many subintervals, and assigning to each of these subintervals one of the indicator numbers $l, l+1, \ldots, n$.

For $\lambda_{l+1}^{(l)}$, subdivide I into $m_l + 1$ subintervals of equal length. To the first (from the left), assign the number l. And to each of the others, assign the number $l+1$.



For $\lambda_{l+2}^{(l)}$, refine the previous subdivision by dividing each subinterval corresponding to $l+1$ as follows. Divide the subinterval into $m_{l+1}+1$ subintervals of equal length. To the first of these (from the left), assign the number $l+1$. To each of the others, assign the number $l+2$.



Continue this process iteratively to fill out the sequence $(\lambda_n^{(l)})$.

claim 1 Each sequence $(\lambda_n^{(l)})$ converges to a continuous loop $\lambda^{(l)}$ based at $*$ in HE.

Proof Let d be the Euclidean metric on HE. Because HE is a compact metric space, the space,

$$C_*(I,HE)=\{\alpha : I \to HE \mid \alpha \text{ is continuous, and } \alpha(0)=\alpha(1)=*\},$$

is complete with respect to the sup metric induced by d; namely,

$$\tilde{d}(\alpha,\beta)=\sup\{d(\alpha(x),\beta(x)) \mid x \in I\}.$$

It suffices, therefore, to show that each sequence $(\lambda_n^{(l)})$ is a Cauchy sequence. So fix a positive integer l, and let $\varepsilon>0$. By construction, if $m,n \in Z^+$ with $m<n$, then

$$\tilde{d}(\lambda_m^{(l)},\lambda_n^{(l)}) \le diam(C_{m-1})=2/(m-1).$$

Thus, taking $M>0$ so that $2/(M-1)<\varepsilon$,

$$\tilde{d}(\lambda_m^{(l)},\lambda_n^{(l)})<\varepsilon \quad \forall m,n \ge M,$$

showing that $(\lambda_n^{(l)})$ is a Cauchy sequence. √

claim 2 For each positive integer l, $\lambda^{(l)} \approx_p \gamma_l (\lambda^{(l+1)})^{m_l}$.

Proof Fix a positive integer l. For each $i = 1,2,\ldots,m_l + 1$, define the linear homeomorphism,

$$h_i : [\frac{i-1}{m_l+1}, \frac{i}{m_l+1}] \to HE : x \to x(m_l+1) - (i-1).$$

For $n > l$,

$$\lambda_n^{(l)}\Big|_{[\frac{i-1}{m_l+1}, \frac{i}{m_l+1}]} = \begin{cases} \gamma_l \circ h_i & \text{if } i = 1 \\ \lambda_n^{(l+1)} \circ h_i & \text{if } i = 2,3,\ldots,m_l+1 \end{cases}.$$

So let $t \in I$. Then $t \in [\frac{i-1}{m_l+1}, \frac{i}{m_l+1}]$ for some $i = 1,2,\ldots,m_l+1$. If $i = 1$, then

$$\lambda^{(l)}(t) = \lim_{n\to\infty} \lambda_n^{(l)}(t) = \lim_{n\to\infty} (\gamma_l \circ h_i)(t) = (\gamma_l \circ h_i)(t).$$

If $i = 2,3,\ldots,m_l+1$, then

$$\lambda^{(l)}(t) = \lim_{n\to\infty} \lambda_n^{(l)}(t) = \lim_{n\to\infty} (\lambda_n^{(l+1)} \circ h_i)(t) = (\lambda^{(l+1)} \circ h_i)(t).$$

Thus, $\lambda^{(l)}$ is pointwise equivalent to a loop in HE which is path-homotopic to $\gamma_l (\lambda^{(l+1)})^{m_l}$. √

For each positive integer l, denote $x^{(l)} = [\lambda^{(l)}] \in G$. We call $\{x^{(l)} \mid l \in Z^+\}$ the **L-Set over** $\sigma = (m_k)$. By claim 2, the elements of this set satisfy the relations

$$x^{(l)} = x_l (x^{(l+1)})^{m_l}.$$

Theorom 2.8 Let $\varphi : G \to Z$ be a homomorphism. Then $\varphi(x_i) = 0$ for all but finitely many of the i.

Proof Suppose to the contrary that for each $N > 0$, there exists $i > N$ such that $\varphi(x_i) \neq 0$. Then we can build a strictly increasing sequence (i_j) with $\varphi(x_{i_j}) \neq 0$ for all j. Since each $x_{i_j}^{\pm 3}$ is supported in $\bigcup_{n \geq i_j} C_n$, there exists by **Lemma 2.7** an endomorphism ϑ of G mapping each x_j to $x_{i_j}^{\pm 3}$, where the sign in the exponent is chosen to be the sign of $\varphi(x_{i_j})$. Identifying φ with the composition $\varphi \circ \vartheta$, we have for each j that

$$\varphi(x_j) = \begin{cases} 3\varphi(x_{i_j}) & \text{if } \varphi(x_{i_j}) > 0 \\ -3\varphi(x_{i_j}) & \text{if } \varphi(x_{i_j}) < 0 \end{cases}.$$

Thus, $\varphi(x_j) \geq 3$ for all j.

Putting $a_j = \varphi(x_j)$, consider the L-set $\{x^{(l)} \mid l \in Z^+\}$ over the sequence (a_j). In accordance with the relations on this set,

$$\varphi(x^{(l)}) = \varphi(x_l(x^{(l+1)})^{a_l}) = \varphi(x_l) + a_l\varphi(x^{(l+1)}) = a_l + a_l\varphi(x^{(l+1)}) .$$

In Particular,

$$\begin{aligned}\varphi(x^{(1)}) &= a_1 + a_1\varphi(x^{(2)}) \\ &= a_1 + a_1(a_2 + a_2\varphi(x^{(3)})) \\ &= a_1 + a_1(a_2 + a_2(a_3 + a_3\varphi(x^{(4)}))) \\ &\vdots \\ &= a_1 + a_1a_2 + \cdots + a_1a_2\cdots a_n + a_1a_2\cdots a_n\varphi(x^{(n+1)}) .\end{aligned}$$

As suggested by De Smit, "the integer $\varphi(\mathrm{x}^{(1)})$ now satisfies congruence conditions that are too strong to hold for any integer." For each $n \geq 2$, put

$$b_n = a_1 + a_1a_2 + \cdots + a_1a_2\cdots a_{n-1}, \quad \text{and} \quad c_n = a_1a_2\cdots a_n.$$

Then for each such n, we may write

$$\varphi(x^{(1)}) = b_n + c_n + c_n\varphi(x^{(n+1)}) = b_n + c_n(1 + \varphi(x^{(n+1)})) . \tag{1}$$

Observe that each $b_n < c_n$; for by induction, $b_2 = a_1 < a_1a_2 = c_2$, and

$$b_n < c_n \;\Rightarrow\; b_n + c_n < 2c_n \;\Rightarrow\; b_{n+1} < 2a_1a_2\cdots a_n < a_1a_2\cdots a_{n+1} = c_{n+1}.$$

Observe also that because $a_n \geq 3$,

$$\begin{aligned} 2c_n < c_{n+1} &\Rightarrow c_n < c_{n+1} - c_n \\ &\Rightarrow c_n - b_n < c_{n+1} - c_n - b_n = c_{n+1} - (c_n + b_n) = c_{n+1} - b_{n+1} ,\end{aligned}$$

so the sequence $(c_n - b_n) \to \infty$.

case 1 Assume $\varphi(x^{(1)}) \geq 0$. Because $(b_n) \to \infty$, there exists n_0 such that $\varphi(x^{(1)}) < b_{n_0}$. But then, $0 \leq \varphi(x^{(1)}) < b_{n_0} < c_{n_0}$,

$$\Rightarrow c_{n_0} > \left|\varphi(x^{(1)}) - b_{n_0}\right| \overset{(1)}{=} \left|c_{n_0}(1 + \varphi(x^{(n_0+1)})\right| = c_{n_0}\left|1 + \varphi(x^{(n_0+1)})\right|,$$

so $\varphi(x^{(n_0+1)}) = -1$. But then by **(1)**, $\varphi(x^{(1)}) = b_{n_0}$, a contradiction.

case 2 Assume $\varphi(x^{(1)}) < b_{n_0}$. Because $(c_n - b_n) \to \infty$, there exists n_0 such that $-\varphi(x^{(1)}) < c_{n_0} - b_{n_0}$,

$$\Rightarrow -c_{n_0} < \varphi(x^{(1)}) - b_{n_0} < 0 \;\Rightarrow c_{n_0} > \left|\varphi(x^{(1)}) - b_{n_0}\right|.$$

But as in case 1, this leads to the fact that $\varphi(x^{(1)}) = b_{n_0}$, contradicting the assumption. √

Lemma 2.9 For each positive integer j, G is generated by its subgroups $G(\{1,\ldots,j\})$ and $G(\{j+1,j+2,\ldots\})$.

Proof Fixing $j \in Z^{+}$, let $x=[\lambda]\in G$, where $\lambda : I \to G$; and put $V=\bigcup_{k=1}^{j}(C_k-\{*\})\overset{op}{\subseteq} HE$. Then $\lambda^{-1}(V)\overset{op}{\subseteq} I$, so there exists a countable set A and pairwise disjoint intervals (a_α,b_α) in I such that

$$\lambda^{-1}(V)=\dot{\bigcup_{\alpha\in A}}(a_\alpha,b_\alpha).$$

One notes that for each $\alpha\in A$, $\lambda(a_\alpha)=\lambda(b_\alpha)=*$, and there exists $k_\alpha\in\{1,\ldots,j\}$ such that $\lambda((a_\alpha,b_\alpha))\subseteq C_{k_\alpha}-\{*\}$. For $k=1,\ldots,j$, select $\xi_k\in C_k-\{*\}$, and put $\Xi=\{\xi_1,\ldots,\xi_j\}$.

Let us show that the collection

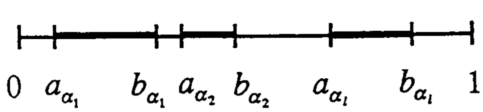
$$\Lambda=\{\alpha\in A \mid (a_\alpha,b_\alpha)\cap\lambda^{-1}(\Xi)\neq\varnothing\}$$

is finite. For each $\alpha\in\Lambda$, select $c_\alpha\in(a_\alpha,b_\alpha)\cap\lambda^{-1}(\Xi)$, and put $C=\{c_\alpha \mid \alpha\in\Lambda\}$. It suffices to show that C is finite. Again, we invoke the fact that closed, discrete subspaces of compact spaces are finite.

First, each $\{c_\alpha\}=(a_\alpha,b_\alpha)\cap C\overset{op}{\subseteq} C$, so C is discrete in I. To see that C is closed, we show that C has no limit points. So suppose that $z\in I$ were a limit point of C, and let (c_{α_n}) be a sequence in C converging to z. Then by continuity of λ, $\lambda(z)=\lim_{n\to\infty}(\lambda(c_{\alpha_n}))\in\Xi$, so $z\in(a_i,b_i)$ for some $i\in\Lambda$. If $z=c_i$, then (a_i,b_i) is a neighborhood of z in I containing no other c_α, a contradiction. If $z\neq c_i$, take $\varepsilon>0$ so that $c_i\notin B_\varepsilon(z)\cap(a_i,b_i)$.

Now, order $\Lambda=\{\alpha_1,\ldots,\alpha_l\}$ so that $a_{\alpha_i}\le a_{\alpha_{i+1}}$ for all $i=1,\ldots,l$. For $k=0,\ldots,l$, put

$$y_k=\begin{cases}[\lambda|_{[0,a_{\alpha_1}]}] & \text{if } k=0\\ [\lambda|_{[b_{\alpha_k},a_{\alpha_{k+1}}]}] & \text{if } k\in\{1,\ldots,l-1\}\\ [\lambda|_{[b_{\alpha_l},1]}] & \text{if } k=l\end{cases},$$



and for $k=1,\ldots,l$, put $w_k=[\lambda|_{[a_{\alpha_k},b_{\alpha_k}]}]$. Then $x=y_0w_1y_1\cdots w_ly_l$, where each $w_k\in G(\{1,\ldots,j\})$, and each $y_k\in\pi_1(HE-\Xi,*)$. Observing that $HE(\{j+1,j+2,\ldots\})$ is a strong deformation retract of $HE-\Xi$, identify each y_k as an element of $G(\{j+1,j+2,\ldots\})$. √

Theorom 2.10 Let $\varphi : G \to Z$ be a homomorphism such that $\varphi(x_i) = 0$ for all i. Then $\varphi(x) = 0$ for all $x \in G$.

Proof Suppose that $\varphi(x) \neq 0$ for some $x \in G$, and fix $j \in Z^+$. By the preceding lemma, we may write $x = y_0 w_1 y_1 \cdots w_l y_l$, where each w_i is an element of the free group $F_j = G(\{1,\ldots,j\})$, and each y_i is an element of $G(\{j, j+1,\ldots\})$. Putting $y = y_0 \cdots y_l$, $\varphi(y) = \varphi(x) \neq 0$, as each $\varphi(w_i) = 0$. By **Lemma 2.7**, there exists an endomorphism of G mapping x_j to y. The composition of this function with φ yields a homomorphism $: G \to Z$ assigning each x_j to a non-zero integer, contradicting **Theorom 2.8.** √

Bibliography

1. H.B. Griffiths, *Infinite Products of Semigroups and Local Connectivity,* Proceedings of the London Mathematical Society, Third Series, Volume 6, 1956

2. J.W. Morgan and I. Morrison, *A Van Kampen Theorem for Weak Joins,* Proceedings of the London Mathematical Society, Third Series, Volume 56, 1986

3. Graham Higman, *Unrestricted Free Products and Varieties of Topological Groups,* The Journal of The London Mathematical Society, Volume 27, 1957

4. Bart De Smit, *The Fundamental Group of the Hawaiian Earring is not Free,* International Journal of Algebra and Computation, Volume 2, Number 1, 1992